Geometric analysis for the metropolis algorithm on Lipschitz domains

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Abstract

This paper gives geometric tools: comparison, Nash and Sobolev inequalities for pieces of the relevent Markov operators, that give useful bounds on rates of convergence for the Metropolis algorithm. As an example, we treat the random placement of N hard discs in the unit square, the original application of the Metropolis algorithm.

Contents

1 Introduction and results **2**

2 A proof of theorem thm1 1.1 **5**

3 A proof of theorem thm2 1.2 **15**

4 Application to random placement of non-overlapping balls **19**

References **31**

1 Introduction and results

Let Ω be a bounded, connected subset of $\mathbb{R}^d$. We assume that its boundary, $\partial\Omega$ has Lipschitz regularity. Let B_1 be the unit ball of $\mathbb{R}^d$ and $\varphi(z) = \frac{1}{Vol(B_1)} 1_{B_1}(z)$ so that $\int \varphi(z)dz = 1$. Let $\rho(x)$ be a measurable positive bounded function on $\overline{\Omega}$ such that $\int_\Omega \rho(x)dx = 1$. For $h \in]0,1]$, set

$$K_{h,\rho}(x,y) = h^{-d}\varphi(\frac{x-y}{h})min(\frac{\rho(y)}{\rho(x)},1) \tag{1.1}$$

and let $T_{h,\rho}$ be the () Metropolis operator associated to these data, that is

$$\begin{aligned} T_{h,\rho}(u)(x) &= m_{h,\rho}(x)u(x) + \int_\Omega K_{h,\rho}(x,y)u(y)dy \\ m_{h,\rho}(x) &= 1 - \int_\Omega K_{h,\rho}(x,y)dy \geq 0 \end{aligned} \tag{1.2}$$

Then the Metropolis kernel $T_{h,\rho}(x,dy) = m_{h,\rho}(x)\delta_{x=y} + K_{h,\rho}(x,y)dy$ is a Markov kernel, the operator $T_{h,\rho}$ is self-adjoint on $L^2(\Omega, \rho(x)dx)$, and thus the probability measure $\rho(x)dx$ on Ω is stationary. For $n \geq 1$, we denote by $T^n_{h,\rho}(x,dy)$ the kernel of the iterate operator $(T_{h,\rho})^n$. For any $x \in \Omega$, $T^n_{h,\rho}(x,dy)$ is a probability measure on Ω, and our main goal is to get some estimates on the rate of convergence, when $n \to +\infty$, of the probability $T^n_{h,\rho}(x,dy)$ toward the stationary probability $\rho(y)dy$.

A good example to keep in mind is the random placement of N non-overlapping discs of radius $\varepsilon > 0$ in the unit square. This was the original motivation for the work of Metropolis et al ([MRR+53]). One version of their algorithm goes as follows: from a feasable configuration, pick a disc (uniformly at random) and a point within distance h of the center of the chosen disc (uniformly at random). If recentering the chosen disc at the chosen point results in a feasable configuration, the change is made. Otherwise, the configuration is kept as it started. If N is fixed and ε and h are small, this gives a Markov chain with a uniform stationary distribution over all feasable configurations. The state space consists of the N centers corresponding to feasible configurations. It is a bounded domain with a Lipshitz boundary (see section 4, proposition 4.1). It is non-convex (because of the non-overlaping constraints). The scientific motivation for the study of random packing of hard discs is clearly described in Uhlenbeck ([?], section 5, pg 18). An overview of the large literature is in Lowen ([?]). Entry to the zoo of modern algorithms to do the simulation (particularly in the dense case) with many examples is in **[German guy, reference coming]**. Further discussion, showing that the problem is still of current interest, is in Radin ([?]).

We shall denote by $g(h,\rho)$ the spectral gap of the Metropolis operator $T_{h,\rho}$. It is defined as the best constant such that the following inequality holds true for all $u \in L^2(\rho) = L^2(\Omega, \rho(x)dx)$

$$\|u\|^2_{L^2(\rho)} - (u|1)^2_{L^2(\rho)} \leq \frac{1}{g(h,\rho)}(u - T_{h,\rho}u|u)_{L^2(\rho)} \tag{1.3}$$

or equivalently

$$\int_{\Omega\times\Omega} |u(x)-u(y)|^2\rho(x)\rho(y)dxdy \leq \frac{1}{g(h,\rho)}\int_{\Omega\times\Omega} K_{h,\rho}(x,y)|u(x)-u(y)|^2\rho(x)dxdy \tag{1.4}$$

def1 **Definition 1.1** *We say that an open set $\Omega \subset \mathbb{R}^d$ is Lipschitz if it is bounded and for all $a \in \partial\Omega$ there exists an orthonormal basis $\mathcal{R}_a$ of $\mathbb{R}^d$, an open set $V = V' \times]-\alpha, \alpha[$ and a Lipschitz map $\eta : V' \to]-\alpha, \alpha[$ such that in the coordinates of $\mathcal{R}_a$, we have*

$$\begin{aligned} V \cap \Omega &= \{(y', y_d < \eta(y')), (y', y_d) \in V' \times]-\alpha, \alpha[\} \\ V \cap \partial\Omega &= \{(y', \eta(y')), y' \in V'\}. \end{aligned} \tag{1.5}$$

Our first result is the following:

thm1 **Theorem 1.1** *Let Ω be an open, connected, bounded and Lipschitz subset of $\mathbb{R}^d$. Let $0 < m \leq M < \infty$ be given numbers. There exists $h_0 > 0$, $\delta_0 \in]0, 1/2[$ and constants $C_i > 0$ such that for any $h \in]0, h_0]$, and any probability density ρ on Ω which satisfies for all x, $m \leq \rho(x) \leq M$, the following holds true.*

i) The spectrum of $T_{h,\rho}$ is a subset of $[-1 + \delta_0, 1]$, 1 is a simple eigenvalue of $T_{h,\rho}$, and $Spec(T_{h,\rho}) \cap [1 - \delta_0, 1]$ is discrete. Moreover, for any $0 \leq \lambda \leq \delta_0 h^{-2}$, the number of eigenvalues of $T_{h,\rho}$ in $[1 - h^2\lambda, 1]$ (with multiplicity) is bounded by $C_1(1 + \lambda)^{d/2}$.

ii) The spectral gap satisfies

$$C_2 h^2 \leq g(h, \rho) \leq C_3 h^2 \tag{1.6}$$ gap3

and the following estimate holds true for all integer n

$$sup_{x \in \Omega} \|T^n_{h,\rho}(x, dy) - \rho(y)dy\|_{TV} \leq C_4 e^{-ng(h,\rho)} \tag{1.7}$$ 1.7

The next result will give some more information on the behavior of the spectral gap $g(h, \rho)$ when $h \to 0$. To state this result, let

$$\alpha_d = \int \varphi(z) z_1^2 dz = \frac{1}{d} \int \varphi(z)|z|^2 dz = \frac{1}{d+2} \tag{1.8}$$ 1.1

and let us define $\nu(\rho)$ as the best constant such that the following inequality holds true for all u in the Sobolev space $H^1(\Omega)$

$$\|u\|^2_{L^2(\rho)} - (u|1)^2_{L^2(\rho)} \leq \frac{1}{\nu(\rho)} \frac{\alpha_d}{2} \int_\Omega |\nabla u|^2(x)\rho(x)dx \tag{1.9}$$ nu1

or equivalently

$$\int_{\Omega \times \Omega} |u(x) - u(y)|^2 \rho(x)\rho(y)dxdy \leq \frac{\alpha_d}{\nu(\rho)} \int_\Omega |\nabla u|^2(x)\rho(x)dx \tag{1.10}$$ nu2

Observe that for a Lipschitz domain Ω, the constant $\nu(\rho)$ is well-defined thanks to Sobolev embedding. For a smooth density ρ, this number $\nu(\rho) > 0$ is strongly related to the unbounded operator L_ρ acting on on $L^2(\rho)$

$$\begin{aligned} L_\rho(u)(x) &= \frac{-\alpha_d}{2}(\triangle u + \frac{\nabla\rho}{\rho}.\nabla u) \\ D(L_\rho) &= \{u \in H^1(\Omega),\ -\Delta u \in L^2(\Omega), \partial_n u|_{\partial\Omega} = 0\} \end{aligned} \tag{1.11}$$ 1.3

When Ω has smooth boundary, standard elliptic regularity results show that for any $u \in H^1(\Omega)$ such that $-\Delta u \in L^2(\Omega)$, the normal derivative of u at the boundary, $\partial_n u = \overrightarrow{n}(x).\nabla u|_{\partial\Omega}$ is well define and belongs to the Sobolev space $H^{1/2}(\Omega)$. Here, we denote by $\overrightarrow{n}(x)$ the incoming unitary normal vector to $\partial\Omega$ at a point x. In the case where $\partial\Omega$ has only Lipschitz regularity, the normal $\overrightarrow{n}(x)$ is well defined for allmost every $x \in \partial\Omega$ (with respect to the measure σ induced on the boundary). Using a suitable covering of Ω it is possible to define a trace operator $\gamma_0 : H^1(\Omega) \mapsto L^2(\partial\Omega)$ which is equal to the usual trace in the case of a smooth boundary. We sometimes denote $\gamma_0(u) = u|_{\partial\Omega}$. The space defined by $H^{1/2}(\partial\Omega) = Ran(\gamma_0)$ doesn't depend on the charts used to define γ_0, and equipped with the norm $\|u\|_{H^{1/2}} = \inf\{\|v\|_{H^1},\ \gamma_0(v) = u\}$ it is a reflexif Banach space. Then, we can set $H^{-1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)^*$ and for $u \in H^{-1/2}(\partial\Omega)$, the support of u can be defined in a standard way. The trace operator acting on vector fields $u \in L^2, div(u) \in L^2$

$$\gamma_1 : \{u \in (L^2(\Omega))^d, div(u) \in L^2(\Omega)\} \to H^{-1/2}(\partial\Omega) \tag{1.12}$$

1.3.2

is then defined by the formula

$$\int_\Omega div(u)(x)v(x)dx = -\int_\Omega u(x).\nabla v(x)dx - \int_{\partial\Omega} \gamma_1(u)v|_{\partial\Omega}d\sigma(x) \tag{1.13}$$

1.3.3

In particular, for $u \in H^1(\Omega)$ satisfying $-\Delta u = div\nabla u \in L^2(\Omega)$, we can define $\partial_n u|_{\partial\Omega} = \gamma_1(\nabla u) \in H^{-1/2}(\partial\Omega)$ and the set $D(L_\rho)$ is well defined. From (1.13) we deduce that for any $u \in H^1(\Omega)$ with $\Delta u \in L^2$ and any $v \in H^1(\Omega)$ we have

$$\langle (L_\rho + 1)u, v\rangle_{L^2(\rho)} = \frac{\alpha_d}{2}\Big(\langle \nabla u, \nabla v\rangle_{L^2(\rho)} + \langle \partial_n u, \rho v\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}\Big) + \langle u, v\rangle_{L^2(\rho)} \tag{1.14}$$

1.3.4

Then, it is standard that L_ρ is the self-adjoint realisation of the Dirichlet form

$$\frac{\alpha_d}{2}\int_\Omega |\nabla u(x)|^2 \rho(x)dx. \tag{1.15}$$

1.4

Sobolev embeddings show that L_ρ has a compact resolvant and we denote its spectrum by $\nu_0 = 0 < \nu_1 < \nu_2 < \ldots$ and by m_j the multiplicity of ν_j. In particular, $\nu(\rho) = \nu_1$. Observe also that $m_0 = 1$ since $KerL$ is spaned by the constant function equal to 1 .

To state our theorem, we need

def2 **Definition 1.2** *Let Ω be a Lipschitz open set of $\mathbb{R}^d$. We say that $\partial\Omega$ is stratified if $\partial\Omega = \Gamma_{reg} \cup \Gamma_{sing}, \Gamma_{reg} \cap \Gamma_{sing} = \emptyset$ with Γ_{reg} reunion of smooth hypersurfaces, relatively open in $\partial\Omega$, and Γ_{sing} a closed subset of $\mathbb{R}^d$ such that*

$$v \in H^{-1/2}(\partial\Omega) \text{ and } support(v) \subset \Gamma_{sing} \Longrightarrow v = 0 \tag{1.16}$$

uniq

Observe that 1.16 is obviously satisfied if $\partial\Omega$ is smooth, since in that case one can take $\Gamma_{sing} = \emptyset$. More generally, if Ω is a Lipschitz open set of $\mathbb{R}^d$ such that $\partial\Omega = \Gamma_{reg} \cup \Gamma_{sing}$, $\Gamma_{reg} \cap \Gamma_{sing} = \emptyset$, where Γ_{reg} is a smooth hypersurface of $\mathbb{R}^d$, relatively open in $\partial\Omega$, and Γ_{sing} a closed subset of $\mathbb{R}^d$ such that $\Gamma_{sing} = \cup_{j\geq 2} S_j$ where the S_j are smooth disjoints submanifolds of $\mathbb{R}^d$ such that

$$codim_{\mathbb{R}^d} S_j \geq j, \quad \cup_{k\geq j} S_k = \overline{S_j} \tag{1.17}$$

then Ω is stratified, since in that case, if $v \in H^{-1/2}(\partial\Omega)$ is such that near a point x_0, the support of v is contained in a submanifold S of codimension ≥ 2 in $\mathbb{R}^d$, then $v = 0$ near x_0. This follows from the fact that S has codimension ≥ 1 in $\partial\Omega$, and if $u \in \mathcal{D}'(\mathbb{R}^p)$ is such that $u \in H^{-1/2}_{loc}(\mathbb{R}^p)$ and $support(u) \subset \{x_1 = 0\}$, then $u = 0$. As an example, a cube in $\mathbb{R}^d$ is stratified.

thm2 **Theorem 1.2** *Let Ω be an open, connected, bounded and Lipschitz subset of $\mathbb{R}^d$, such that $\partial\Omega$ is stratified. Assume that the positive density ρ is continuous on $\overline{\Omega}$. Then*

$$lim_{h\to 0} h^{-2} g(h,\rho) = \nu(\rho) \tag{1.18}$$ 1.6

Moreover, if the density ρ is smooth up to the regular part Γ_{reg} of the boundary $\partial\Omega$, then for any $R > 0$ and $\varepsilon > 0$ such that $\nu_{j+1} - \nu_j > 2\varepsilon$ for $\nu_{j+2} < R$, there exists $h_1 > 0$ such that one has for all $h \in]0, h_1]$

$$Spec(\frac{1 - T_{h,\rho}}{h^2}) \cap]0, R] \subset \cup_{j\geq 1} [\nu_j - \varepsilon, \nu_j + \varepsilon] \tag{1.19}$$ 1.5

and the number of eigenvalues of $\frac{1-T_{h,\rho}}{h^2}$ in the interval $[\nu_j - \varepsilon, \nu_j + \varepsilon]$ is equal to m_j.

Theorem 1.1 is proved in section 2. This is done from the spectrum of the operator by comparison with a 'ball walk' on a big box B containing Ω. One novelty is the use of 'normal extensions' of functions from Ω to B. When the Dirichlet forms and stationary distributions for random walk on a compact group are comparible, the rates of convergence are comparable as well ([?], **lemma ??**). Here, the Metropolis Markov chain is far from a random walk on a group. Indeed, because of the holding implicit in the Metropolis algorithm, the operator doesnt have any smoothing properties. The transfer of information is carried out by a Sobolev inequality for a spectraly truncated part of the operator. This is transfered to a Nash inequality and then an inductive argument of Hebisch (**see also**[ ]) is used to get decay bounds on iterates of the kernel. A further technique is the use of crude Weyl type estimates to get bounds on the number of eigenvalues close to 1. All of these enter the proof of the total variation estimate 1.7. All of these techniques seem broadly applicable. Theorem 1.2 is proved in section 3. It gives rigourous underpinnings to a general picture of the spectrum of the Metropolis algorithm based on small steps. This was observed and proved in special cases ([?], [DL07]). The picture is this: because of the holding (or presence of the multiplier $m_{h,\rho}$ in 1.2) in the Metropolis algorithm, the operator always has continuous spectrum. This is well isolated from 1 and can be neglected in bounding rates of convergence. The spectrum near 1 is discrete and for h small, merges with the spectrum of an associated Neumann problem. This is an analytic version of the convergence of the discrete time Metropolis chain to the Langevin diffusion with generator 1.11. See Lepingle ([?]) **Paper of Burdzy and Chen reference coming** COAUTHORS SAY A SENTENCE OR TWO ABOUT THE IDEA OF PROOF? In section 4, we return to the hard disc problem showing that the operators and domains involved satisfy our hypothesis. Precisely, in theorem 4.1 we shall prove that the results of theorem 1.1 and theorem 1.2 holds true in this case.

2 A proof of theorem 1.1

sec2

Let us recall that

$$K_{h,\rho}(x,y) = h^{-d}\varphi(\frac{x-y}{h}) min(\frac{\rho(y)}{\rho(x)}, 1) \tag{2.1}$$

so that

$$
\begin{aligned}
&T_{h,\rho}(u) = u - Q_{h,\rho}(u) \\
&Q_{h,\rho}(u)(x) = \int_\Omega K_{h,\rho}(x,y)(u(x)-u(y))dy \\
&((1-T_{h,\rho})u|u)_{L^2(\rho)} = \frac{1}{2}\int\int_{\Omega\times\Omega} |u(x)-u(y)|^2 K_{h,\rho}(x,y)\rho(x)dxdy
\end{aligned}
\tag{2.2}
$$

For the proof of theorem 1.1, we will not really care on the precise choice of the density ρ. In fact, if ρ_1, ρ_2 are two densities such that $m \leq \rho_i(x) \leq M$ for all x, then

$$
\begin{aligned}
&\rho_2(x) \leq \rho_1(x)(1 + \frac{\|\rho_1-\rho_2\|_\infty}{m}) \\
&K_{h,\rho_1}(x,y)\rho_1(x) \leq K_{h,\rho_2}(x,y)\rho_2(x)(1+\frac{\|\rho_1-\rho_2\|_\infty}{m})
\end{aligned}
\tag{2.3}
$$

and this implies using the definition 1.4 of the spectral gap and of ν_ρ

$$
\begin{aligned}
&\frac{g_{h,\rho_1}}{g_{h,\rho_2}} \leq (1+\frac{\|\rho_1-\rho_2\|_\infty}{m})^3 \\
&\frac{\nu_{\rho_1}}{\nu_{\rho_2}} \leq (1+\frac{\|\rho_1-\rho_2\|_\infty}{m})^3
\end{aligned}
\tag{2.4}
$$

In particular, it is sufficient to prove 1.6 for a constant density.

Observe that since Ω is Lipschitz, from 1.2 and 2.3, there exists $h_0 > 0, \delta_0 > 0$ such that for any density ρ with $m \leq \rho(x) \leq M$ one has $sup_{x\in\Omega} m_{h,\rho}(x) \leq 1-2\delta_0$. Thus the essential spectrum of T_h is a subset of $[0, 1-2\delta_0]$. The proof that for some $\delta_0 > 0$, independent of ρ, one has $Spec(T_{h,\rho}) \subset [-1+\delta_0, 1]$ for all $h \in]0, h_0]$ is the following: one has

$$
(u + T_{h,\rho}u|u)_{L^2(\rho)} = \frac{1}{2}\int_{\Omega\times\Omega} K_{h,\rho}(x,y)|u(x)+u(y)|^2\rho(x)dxdy + 2(m_{h,\rho}u|u)_{L^2(\rho)}
\tag{2.5}
$$

Therefore, it is sufficient to prove that there exists $h_0, C_0 > 0$ such that the following inequality holds true for all $h \in]0, h_0]$ and all $u \in L^2(\Omega)$

$$
\int_{\Omega\times\Omega} h^{-d}\varphi(\frac{x-y}{h})|u(x)+u(y)|^2 dxdy \geq C_0\|u\|^2_{L^2(\Omega)}
\tag{2.6}
$$

Let $\omega_j \subset \Omega$, $\cup_j \omega_j = \Omega$ be a covering of Ω such that $diam(\omega_j) < h$ and for some $C_i > 0$ independent of h, $vol(\omega_j) \geq C_1 h^d$, and for any j, the number of k such that $\omega_j \cap \omega_k \neq \emptyset$ is less than C_2. Such a covering exists as Ω is Lipschitz. Then

$$
\begin{aligned}
&C_2\int_{\Omega\times\Omega} h^{-d}\varphi(\frac{x-y}{h})|u(x)+u(y)|^2dxdy \geq \sum_j \int_{\omega_j\times\omega_j} h^{-d}\varphi(\frac{x-y}{h})|u(x)+u(y)|^2dxdy \\
&\geq \sum_j h^{-d}\frac{1}{|B_1|}\int_{\omega_j\times\omega_j}|u(x)+u(y)|^2dxdy \geq \sum_j 2h^{-d}\frac{1}{|B_1|}vol(\omega_j)\|u\|^2_{L^2(\omega_j)} \geq \frac{2C_1}{|B_1|}\|u\|^2_{L^2(\Omega)}
\end{aligned}
\tag{2.7}
$$

From 2.7, we get that 2.6 holds true.

For the proof of (1.6) we need a suitable covering of Ω. Given $\epsilon > 0$ small enough, there exists some open sets $\Omega_0, \ldots, \Omega_N$ such that $\{x \in \mathbb{R}^d, dist(x, \overline{\Omega}) \leq \epsilon^2\} \subset \cup_{j=0}^N \Omega_j$, where the Ω_j's have the following properties:

1. $\Omega_0 = \{x \in \Omega, d(x, \partial\Omega) > \epsilon^2\}$

2. For $j = 1, \dots, N$, there exists $r_j > 0$, an affine isometry R_j of $\mathbb{R}^d$ and a Lipschitz map $\varphi_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, denoting $\tilde{\phi}_j(x', x_d) = (x', x_d + \varphi_j(x'))$ and $\phi_j = R_j \circ \tilde{\phi}_j$, we have

$$\begin{aligned} &\phi_j \text{ is injective on } B(0, 2r_j) \times]-2\epsilon, 2\epsilon[\\ &\Omega_j = \phi_j(B(0, r_j) \times]-\epsilon, \epsilon[) \\ &\Omega_j \cap \Omega = \phi_j(B(0, r_j) \times]0, \epsilon[) \\ &\phi_j(B(0, 2r_j) \times]0, 2\epsilon[) \subset \Omega \end{aligned} \tag{2.8}$$ recouv

We put our open set Ω in a large box $B =]-A/2, A/2[^d$ and for $j = 0, \dots, N$ we let $\chi_j \in C_0^\infty(\Omega_j)$ be such that $\sum_j \chi_j(x) = 1$ for $dist(x, \overline{\Omega}) \leq \epsilon^2$. For any function $u \in L^2(\Omega)$, let $u_j, j = 0, \dots, N$ be defined in a neighbourhood of Ω_j by $u_j = u \circ \phi_j \circ S \circ \phi_j^{-1}$, where $S(x', x_d) = (x', -x_d)$ if $x_d < 0$ and $S(x', x_d) = (x', x_d)$ if $x_d \geq 0$. For $x \in \Omega \cap \Omega_j$, one has $u_j(x) = u(x)$ and we define

$$E(u)(x) = \sum_{j=0}^{N} \chi_j(x) u_j(x) \tag{2.9}$$ 2.3

We observe that $\tilde{\phi}_j^{-1}(x) = (x', x_d - \varphi_j(x'))$. Consequently, as φ_j is Lipschitz-continuous, then ϕ_j and ϕ_j^{-1} are also Lipschitz-continuous. Hence, formula (2.9), gives us an extension map from $L^2(\Omega)$ into $L^2(B)$, which is also bounded from $H^1(\Omega)$ into $H^1(B)$. For $u \in L^2(\Omega), v \in L^2(B)$, set

$$\begin{aligned} &\mathcal{E}_{h,\rho}(u) = ((1 - T_{h,\rho})u|u)_{L^2(\rho)} \\ &\mathcal{E}_h(v) = \int\int_{B \times B, |x-y| \leq h} h^{-d} |v(x) - v(y)|^2 dx dy \end{aligned} \tag{2.10}$$ 2.4

Since for A large, $E(u)$ vanishes near the boundary of B, we can extend $v = E(u)$ as a A-periodic function on $\mathbb{R}^d$, and write its Fourier serie $v(x) = E(u)(x) = \sum_{k \in \mathbb{Z}^d} c_k(v) e^{2i\pi kx/A}$ with $c_k(v) = A^{-d} \int_B e^{-2i\pi kx/A} v(x) dx$. Then

$$\begin{aligned} &\|E(u)\|^2_{L^2(B)} = A^d \sum_k |c_k|^2 \simeq \|u\|^2_{L^2(\Omega)} \\ &\|E(u)\|^2_{H^1(B)} = A^d \sum_k (1 + 4\pi^2 k^2/A^2)|c_k|^2 \simeq \|u\|^2_{H^1(\Omega)} \end{aligned} \tag{2.11}$$ 2.5

Moreover, one gets

$$\begin{aligned} &\mathcal{E}_h(v) = A^d \sum_k |c_k|^2 \theta(hk) \\ &\theta(\xi) = \int_{|z| \leq 1} |e^{2i\pi\xi z/A} - 1|^2 dz \end{aligned} \tag{2.12}$$ 2.6

Observe that the function θ is non-negative, quadratic near 0 and has a positive lower bound for $|\xi| \geq 1$.

lem0 **Lemma 2.1** *For all $\alpha > 1$, there exists $C > 0$ and $h_0 > 0$ such that*

$$\forall u \in L^2(\Omega), \forall h \in]0, h_0], \ \mathcal{E}_{\alpha h,\rho}(u) \leq C \mathcal{E}_{h,\rho}(u). \tag{2.13}$$

Proof. Using 2.2 and 2.3, we observe that it suffices to prove the lemma in the case where $\rho(x) = \rho$ is constant, and we first we show the result when Ω is convex. In that case, since $|u(x) - u(y)| \le |u(x) - u(\frac{x+y}{2})| + |u(\frac{x+y}{2}) - u(y)|$, one has

$$\begin{aligned}
\mathcal{E}_{\alpha h,\rho}(u) &= \frac{(h\alpha)^{-d}}{2Vol(B_1)} \int_\Omega \int_\Omega 1_{|x-y|\le \alpha h} |u(x) - u(y)|^2 \rho dx dy \\
&\le \frac{(h\alpha)^{-d}}{Vol(B_1)} \int_\Omega \int_\Omega 1_{|x-y|\le \alpha h} |u(x) - u(\frac{x+y}{2})|^2 \rho dx dy \qquad (2.14)\\
&\le \frac{2^{2-d}(h\alpha/2)^{-d}}{Vol(B_1)} \int_{\phi(\Omega\times\Omega)} 1_{|x-y|\le \frac{\alpha h}{2}} |u(x) - u(y)|^2 \rho dx dy
\end{aligned}$$

where $\phi(x,y) = (x, \frac{x+y}{2})$. As Ω is convex $\phi(\Omega\times\Omega) \subset \Omega\times\Omega$ and we get $\mathcal{E}_{\alpha h,\rho}(u) \le 2\mathcal{E}_{\frac{\alpha h}{2},\rho}(u)$. Iterating this process we obtain the anounced result for convex domains.

In the general case, we use the local covering introduced in 2.8. Let $\Omega_i^+ = \Omega_i \cap \Omega$ (respectively $\Omega_i^- = \Omega_i \cap (\mathbb{R}^d \setminus \Omega)$) and $U_i(h) = \{(x,y) \in \Omega_i^+ \times \Omega, |x-y| \le \alpha h\}$. Since by 2.2, $\Omega \subset \cup_i \Omega_i^+$, we have $\mathcal{E}_{\alpha h,\rho}(u) \le \sum_{i=0}^N \mathcal{E}^i_{\alpha h,\rho}(u)$ with

$$\mathcal{E}^i_{\alpha h,\rho}(u) = \frac{(\alpha h)^{-d}}{2Vol(B_1)} \int_{U_i(h)} 1_{|x-y|\le \alpha h} |u(x) - u(y)|^2 \rho dx dy. \qquad (2.15)$$

Let us estimate $\mathcal{E}^0_{\alpha h,\rho}(u)$. For $h \in]0, \epsilon^2/\alpha[$ and $(x,y) \in U_0(h)$, we have $[x,y] \subset \Omega$. Therefore, the change of variable $\phi(x,y) = (x, \frac{x+y}{2})$ maps $U_0(h)$ into $\Omega_0 \times \Omega$ and we get as above

$$\mathcal{E}^0_{\alpha h,\rho}(u) \le \frac{(\alpha h)^{-d}}{Vol(B_1)} \int_{U_0(h)} 1_{|x-y|\le \alpha h} |u(x) - u(\frac{x+y}{2})|^2 \rho dx dy \le 2\mathcal{E}_{\frac{\alpha h}{2},\rho}(u). \qquad (2.16)$$

For $i \ne 0$ and $h > 0$ small enough, we remark that $U_i(h) \subset \tilde{\Omega}_i^+ \times \tilde{\Omega}_i^+$, where $\tilde{\Omega}_i^{\pm} = \phi_i(B(0,2r_i) \times \{0 < \pm x_d < 2\epsilon\})$. Denoting $Q_i = B(0,r_i) \times]0,\epsilon[$, $\tilde{Q}_i = B(0,2r_i) \times]0,2\epsilon[$, we can use the Lipschitz-continuous change of variable $\phi_i : \tilde{Q}_i \to \tilde{\Omega}_i^+ \subset \Omega$ to get

$$\mathcal{E}^i_{\alpha h,\rho}(u) \le \frac{(\alpha h)^{-d}}{2Vol(B_1)} \int_{\tilde{Q}_i} \int_{\tilde{Q}_i} J_{\phi_i}(x) J_{\phi_i}(y) 1_{|\phi_i(x)-\phi_i(y)|\le \alpha h} |u\circ\phi_i(x) - u\circ\phi_i(y)|^2 \rho dx dy \quad (2.17)$$

where the Jacobian J_{ϕ_i} of ϕ_i is a bounded function defined almost everywhere. As both ϕ_i, ϕ_i^{-1} are Lipschitz-continuous, there exists $M_i, m_i > 0$ such that for all $x, y \in \tilde{Q}_i$ we have $m_i|x-y| \le |\phi_i(x) - \phi_i(y)| \le M_i|x-y|$. Therefore,

$$\mathcal{E}^i_{\alpha h,\rho}(u) \le Ch^{-d} \int_{\tilde{Q}_i} \int_{\tilde{Q}_i} 1_{|x-y|\le \frac{\alpha h}{m_i}} |u\circ\phi_i(x) - u\circ\phi_i(y)|^2 \rho dx dy, \qquad (2.18)$$

where C denotes a positive constant changing from line to line. As $\tilde{Q}_i$ is convex, it follows from the study of the convex case that

$$\begin{aligned}
\mathcal{E}^i_{\alpha h,\rho}(u) &\le Ch^{-d} \int_{\tilde{Q}_i} \int_{\tilde{Q}_i} 1_{|x-y|\le \frac{h}{M_i}} |u\circ\phi_i(x) - u\circ\phi_i(y)|^2 \rho dx dy \\
&\le Ch^{-d} \int_{\tilde{Q}_i} \int_{\tilde{Q}_i} 1_{|\phi_i(x)-\phi_i(y)|\le h} |u\circ\phi_i(x) - u\circ\phi_i(y)|^2 dx dy \rho \qquad (2.19)\\
&\le Ch^{-d} \int_{\tilde{\Omega}_i^+} \int_{\tilde{\Omega}_i^+} 1_{|x-y|\le h} |u(x) - u(y)|^2 \rho dx dy \le C_i \mathcal{E}_{h,\rho}(u)
\end{aligned}$$

and the proof is complete. □

lem1 **Lemma 2.2** *There exist $C_0, h_0 > 0$ such that the following holds true for any $h \in]0, h_0]$ and any $u \in L^2(\rho)$*

$$\mathcal{E}_{h,\rho}(u)/C_0 \leq \mathcal{E}_h(E(u)) \leq C_0(\mathcal{E}_{h,\rho}(u) + h^2\|u\|_{L^2}^2) \tag{2.20}$$ 2.7

As a byproduct, there exists C_1 such that for all $h \in]0, h_0]$, any function $u \in L^2(\rho)$ such that

$$\|u\|_{L^2(\rho)}^2 + h^{-2}((1 - T_{h,\rho})u|u)_{L^2(\rho)} \leq 1$$

admits a decomposition $u = u_L + u_H$ with $u_L \in H^1(\Omega)$, $\|u_L\|_{H^1} \leq C_1$, and $\|u_H\|_{L^2} \leq C_1 h$.

Proof. Using the second line of 2.3, we may assume that the density ρ is constant. The proof of the left inequality in 2.20 is obvious. For the upper bound, we remark that there exists $C > 0$ such that $\mathcal{E}_h(E(u)) \leq C \sum_{j=0}^{N}(\mathcal{E}_h^{j,1} + \mathcal{E}_h^{j,2})$ with

$$\mathcal{E}_h^{j,1} = h^{-d} \int_{B \times B} 1_{|x-y| \leq h} |\chi_j(x) - \chi_j(y)|^2 |u_j(x)|^2 dxdy \tag{2.21}$$

and

$$\mathcal{E}_h^{j,2} = h^{-d} \int_{B \times B} 1_{|x-y| \leq h} |\chi_j(y)|^2 |u_j(x) - u_j(y)|^2 dxdy \tag{2.22}$$

As the functions χ_j are regular, there exist some $\tilde{\chi}_j \in C_0^\infty(B)$ equal to 1 near the support of χ_j such that

$$\mathcal{E}_h^{j,1} \leq Ch^{-d} \int_B \tilde{\chi}_j(x)|u_j(x)|^2 (\int_B 1_{|x-y| \leq h}|x - y|^2 |dy) dx \leq Ch^2 \|u\|_{L^2(\Omega)}^2 \tag{2.23}$$

In order to estimate $\mathcal{E}_h^{j,2}$ one has to estimate the contribution of the points $x \in \Omega, y \notin \Omega$ and $x \notin \Omega, y \notin \Omega$. All the terms are treated in the same way and we only examine

$$\begin{aligned} \mathcal{E}_h^{j,3} &= h^{-d} \int_{\Omega \times (B \setminus \Omega)} 1_{|x-y| \leq h} |\chi_j(y)|^2 |u_j(x) - u_j(y)|^2 dxdy \\ &= h^{-d} \int_{\tilde{\Omega}_j^+ \times \Omega_j^-} 1_{|x-y| \leq h} |\chi_j(y)|^2 |u(x) - u \circ \phi_j \circ S \circ \phi_j^{-1}(y)|^2 dxdy \end{aligned} \tag{2.24}$$

Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ be the symetry with respect to $\{y_d = 0\}$, so that $S\sigma = Id$ on $\{y_d < 0\}$. We use the Lipschitz-continuous change of variable $\psi_j : y \in \Omega_j^+ \mapsto \phi_j \circ \sigma \circ \phi_j^{-1}(y) \in \Omega_j^-$ to get

$$\mathcal{E}_h^{j,3} \leq Ch^{-d} \int_{\tilde{\Omega}_j^+ \times \Omega_j^+} 1_{|x-\psi_j(y)| \leq h} |\chi_j \circ \psi_j(y)|^2 |u(x) - u(y)|^2 dxdy \tag{2.25}$$

We claim that there exists $\beta > 0$ such that,

$$\forall (x, y) \in \tilde{\Omega}_j^+ \times \Omega_j^+, \ |\psi_j(y) - x| \geq \beta^{-1}|x - y|. \tag{2.26}$$ 2.8

Inded, as both ϕ_j and ϕ_j^{-1} are Lipschitz-continuous, 2.26 is equivalent to find $\beta > 0$ such that

$$\forall (x, y) \in \tilde{\Omega}_j^+ \times \Omega_j^+, \ |\sigma(x) - y| \geq \beta^{-1}|x - y| \tag{2.27}$$

wich is obvious with $\beta = 1$. From (2.26) it follows that for some $\alpha > 1$, one has

$$\mathcal{E}_h^{j,3} \leq Ch^{-d} \int_{\tilde{\Omega}_j^+ \times \Omega_j^+} 1_{|x-y| \leq \alpha h} |u(z) - u(y)|^2 dz dy \leq C\mathcal{E}_{\alpha h, \rho}(u) \tag{2.28}$$

and the upper bound is then a straightforward consequence of Lemma 2.1.

The byproduct is obtain by projecting the extension $v = E(u)$ on low frequencies $h|k| \leq 1$ and high frequencies $h|k| > 1$ and the fact that the function θ is quadratic near 0 and has a positive lower bound for $|\xi| \geq 1$. The proof of lemma 2.2 is complete.

$\square$

We are in position to prove the estimate (1.6) on the spectral gap. To show the right inequality, it suffices to plug a function $u \in C_0^\infty(\Omega)$ into (1.3) with support contained in a small ball $Q \subset \Omega$ and such that $\int_\Omega u(x)\rho(x)dx = 0$. As Q is convex, it follows from Taylor formula that for such u, we have $\langle u - T_h u, u\rangle = O(h^2)$.

To show the left inequality in (1.6) we first observe that it is clearly satisfied when Ω is convex. Indeed, given $u \in L^2(\Omega)$ we have

$$\|u\|^2_{L^2(\Omega)} - \langle u, 1\rangle^2 \leq Ch^{-1} \sum_{k=0}^{K(h)} \int_{\Omega\times\Omega} (u(x + kh(y-x)/\alpha) - u(x + (k+1)h(y-x)/\alpha))^2 dx dy \tag{2.29}$$

where $\alpha = diam(\Omega)$ and $K(h) = O(h^{-1})$. With the new variables $x' = x + kh(y-x)/\alpha$, $y' = x + (k+1)h(y-x)/\alpha$ it comes

$$\|u\|^2_{L^2(\Omega)} - \langle u, 1\rangle^2 \leq C\alpha h^{-d-1} K(h) \int_{\Omega\times\Omega} 1_{|x'-y'|<h} (u(x') - u(y'))^2 dx' dy' \tag{2.30}$$

which proves the left inequality in (1.6) in the case where Ω is convex.

In the general case, we can find some open sets contained in Ω, $\omega_j \subset\subset \Omega_j^+ \subset\subset \tilde{\Omega}_j^+$, $j = 1, \dots, N+M$ such that for $j = 1, \dots, N$, $\Omega_j^+, \tilde{\Omega}_j^+$ are given in the previous lemma, $(\Omega_j^+)_{j=N+1,\dots,N+M}$ are convex $\Omega_0 \subset \cup_{j=N+1}^M \Omega_j^+$, $\Omega \subset \cup_{j=1}^{N+M} \omega_j$, and where $A \subset\subset B$ that $\overline{A}^\Omega \subset B$. Hence for $h > 0$ small enough

$$\begin{aligned} \mathcal{E}_{h,\rho}(u) &\geq C \sum_{j=1}^{N+M} h^{-d} \int_{\Omega_j^+ \times \tilde{\Omega}_j^+} 1_{|x-y|<h} (u(x) - u(y))^2 dx dy \\ &\geq C \sum_{j=1}^{N} h^{-d} \int_{Q_j \times \tilde{Q}_j} 1_{|\phi_j(x) - \phi_j(y)|<h} (u\circ\phi_j(x) - u\circ\phi_j(y))^2 dx dy \\ &\quad + C \sum_{j=N+1}^{N+M} h^{-d} \int_{\Omega_j^+ \times \tilde{\Omega}_j^+} 1_{|x-y|<h} (u(x) - u(y))^2 dx dy \end{aligned} \tag{2.31}$$

From the estimate proved precedently in the convex case, we know that there exists $a > 0$ independant on h such that the second sum in (2.31) is bounded from below by

$$Ch^2 \sum_{j=N+1}^{N+M} \int_{\omega_j \times \Omega_j^+} (u(x) - u(y))^2 dx dy \geq Ch^2 \sum_{j=N+1}^{N+M} \int_{\omega_j \times \Omega, |x-y|<a} (u(x) - u(y))^2 dx dy. \tag{2.32}$$

On the other hand, thanks to the fact that ϕ_j is Lipschitz diffeomorphism, there exists $\alpha > 0$ such that $1_{x-y|<h/\alpha} \leq 1_{|\phi_j(x)-\phi_j(y)|<h} \leq 1_{|x-y|<\alpha h}$. Using the convexity of Q_i and Lemma 2.1 it follows that the first sum in the right hand side of (2.31) is bounded from below by

$$Ch^2 \sum_{j=1}^{N} \int_{\omega_j \times \Omega, |x-y|<a} (u(x) - u(y))^2 dxdy \tag{2.33}$$

Combining (2.31),(2.32) and (2.33), we get

$$\mathcal{E}_{h,\rho}(u) \geq Ch^2 \int_{\Omega\times\Omega,|x-y|<a} (u(x) - u(y))^2 dxdy \tag{2.34}$$

for some fixed $a > 0$ independant on h. Using Lemma 2.1 with $h = a$ we achieve the proof of (1.6).

Lemma 2.3 *There exists $\delta_0 \in]0, 1/2[$ such that $Spec(T_{h,\rho}) \cap [1 - \delta_0, 1]$ is discrete, and for any $0 \leq \lambda \leq \delta_0/h^2$, the number of eigenvalues of T_h in $[1 - h^2\lambda, 1]$ (with multiplicity) is bounded by $C_1(1 + \lambda)^{d/2}$. Moreover, any eigenfuntion $T_h(u) = \lambda u$ with $\lambda \in [1 - \delta_0, 1]$ satisfies the bound*

$$\|u\|_{L^\infty} \leq C_2 h^{-d/2} \|u\|_{L^2} \tag{2.35}$$

Proof. To get 2.35, we just write that since λ is not in the range of m_h, one has

$$u(x) = \frac{1}{\lambda - m_h(x)} \int_\Omega h^{-d} \varphi(\frac{x-y}{h}) min(\frac{\rho(y)}{\rho(x)}, 1) u(y) dy$$

and we apply Cauchy-Schwarz. The important point here is the estimate on the number of eigenvalues in $[1 - h^2\lambda, 1]$ by a power of λ. This is obtain by the min-max and uses 2.20. The min-max gives: if for some closed subspace F of $L^2(\rho)$ with $codim(F) = N$ one has for all $u \in F$, $h^{-2}((1 - T_h)u|u)_{L^2(\rho)} \geq \lambda \|u\|^2_{L^2(\rho)}$, then the number of eigenvalues of T_h in $[1 - h^2\lambda, 1]$ (with multiplicity) is bounded by $codim(F) = N$. Then, we fix $c > 0$ small enough, and we choose for F the subspace of functions u such that their extension $v = E(u)$ is such that the Fourier coefficients satisfy $c_k(E(u)) = 0$ for $|k| \leq D$ with $hD \leq c$. The codimension of this space F is exactly the number of $k \in \mathbb{Z}^d$ such that $|k| \leq D$, since if p is a trigonometric polynomial such that $E^*(p) = 0$, we will have $\int_\Omega p(x)u(x)dx = 0$ for any function u with compact support in Ω and such that $E(u) = u$, and this implies $p = 0$. Thus $codim(F) \simeq (1 + D)^d$. On the other hand, the right inequality in 2.20 gives for $u \in F$, $h^{-2}((1-T_h)u|u)_{L^2(\rho)} \geq C_0(D^2 - C_1)\|u\|^2_{L^2(\rho)}$ for universal C_0, C_1, since by 2.12, there exists $C_0 > 0$ such that one has $\theta(hk)h^{-2} \geq C_0 D^2$ for all $D \leq c/h$ and all $|k| > D$. The proof of our lemma is complete. □

We are now ready to prove the total variation estimate 1.7. Let Π_0 be the orthogonal projector in $L^2(f)$ on the space of constant functions

$$\Pi_0(u)(x) = 1_\Omega(x) \int_\Omega u(y)\rho(y)dy \tag{2.36}$$

Then

$$2sup_{x_0\in\Omega} \|T^n_{h,x_0} - \rho(x)dx\|_{TV} = \|T^n_h - \Pi_0\|_{L^\infty \to L^\infty} \tag{2.37}$$

Thus, we have to prove that there exist C_0, h_0, such that for any n and any $h \in]0, h_0]$, one has

$$\|T_h^n - \Pi_0\|_{L^\infty \to L^\infty} \le C_0 e^{-n g_{h,\rho}} \tag{2.38}$$

Observe that since we know that for h_0 small, the estimate 1.6 holds true for any ρ, we may assume $n \ge Ch^{-2}$. In order to prove 2.38, we split T_h in 3 pieces, according to the spectral theory.

Let $0 < \lambda_{1,h} \le ... \le \lambda_{j,h} \le \lambda_{j+1,h} \le ... \le h^{-2}\delta_0$ be such that the eigenvalues of T_h in the interval $[1-\delta_0, 1[$ are the $1 - h^2\lambda_{j,h}$, with associated orthonormals eigenfunctions $e_{j,h}$

$$T_h(e_{j,h}) = (1 - h^2\lambda_{j,h})e_{j,h}, \quad (e_{j,h}|e_{k,h})_{L^2(\rho)} = \delta_{j,k} \tag{2.39}$$

Then we write $T_h - \Pi_0 = T_{h,1} + T_{h,2} + T_{h,3}$ with

$$\begin{aligned} T_{h,1}(x,y) &= \sum_{\lambda_{1,h} \le \lambda_{j,h} \le h^{-\alpha}} (1 - h^2\lambda_{j,h})e_{j,h}(x)e_{j,h}(y) \\ T_{h,2}(x,y) &= \sum_{h^{-\alpha} < \lambda_{j,h} \le h^{-2}\delta_0} (1 - h^2\lambda_{j,h})e_{j,h}(x)e_{j,h}(y) \\ T_{h,3} &= T_h - \Pi_0 - T_{h,1} - T_{h,2} \end{aligned} \tag{2.40}$$

Here $\alpha > 0$ is a small constant that will be chosen later. One has $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n + T_{h,3}^n$, and we will get the bound 2.38 for each of the 3 terms. We start by very rough bounds. Since there is at most Ch^{-d} eigenvalues $\lambda_{j,h}$ and using the bound (2.35), we get that there exists C independent of $n \ge 1$ and h such that

$$\|T_{1,h}^n\|_{L^\infty \to L^\infty} + \|T_{2,h}^n\|_{L^\infty \to L^\infty} \le Ch^{-3d/2} \tag{2.41}$$

Since T_h^n is bounded by 1 on L^∞, we get from $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n + T_{h,3}^n$

$$\|T_{3,h}^n\|_{L^\infty \to L^\infty} \le Ch^{-3d/2} \tag{2.42}$$

Next we use 1.2 to write $T_h = m_h + R_h$ with

$$\begin{aligned} \|m_h\|_{L^\infty \to L^\infty} &\le \gamma < 1 \\ \|R_h\|_{L^2 \to L^\infty} &\le C_0 h^{-d/2} \end{aligned} \tag{2.43}$$

From this, we deduce that for any $p = 1, 2, ...$, one has $T_h^p = A_{p,h} + B_{p,h}$, with $A_{1,h} = m_h, B_{1,h} = R_h$ and the recurrence relation $A_{p+1,h} = m_h A_{p,h}, B_{p+1,h} = m_h B_{p,h} + R_h T_h^p$. Thus one gets since T_h^p is bounded by 1 on L^2

$$\begin{aligned} \|A_{p,h}\|_{L^\infty \to L^\infty} &\le \gamma^p \\ \|B_{p,h}\|_{L^2 \to L^\infty} &\le C_0 h^{-d/2}(1 + \gamma + ... + \gamma^p) \le C_0 h^{-d/2}/(1-\gamma) \end{aligned} \tag{2.44}$$

Let $\theta = 1 - \delta_0 < 1$ so that $\|T_{3,h}\|_{L^2 \to L^2} \le \theta$. Then one has

$$\|T_{3,h}^n\|_{L^\infty \to L^2} \le \|T_{3,h}^n\|_{L^2 \to L^2} \le \theta^n$$

and for $n \ge 1, p \ge 1$, one gets using 2.44 and 2.42

$$\begin{aligned}
\|T^{p+n}_{3,h}\|_{L^\infty\to L^\infty} &= \|T^p_h T^n_{3,h}\|_{L^\infty\to L^\infty} \\
&\le \|A_{p,h}T^n_{3,h}\|_{L^\infty\to L^\infty} + \|B_{p,h}T^n_{3,h}\|_{L^\infty\to L^\infty} \\
&\le Ch^{-3d/2}\gamma^p + C_0 h^{-d/2}\theta^n/(1-\gamma)
\end{aligned} \tag{2.45}$$

Thus we get for some $C > 0, \mu > 0$,

$$\|T^n_{3,h}\|_{L^\infty\to L^\infty} \le Ce^{-\mu n}, \quad \forall h, \quad \forall n \ge 1/h \tag{2.46}$$

and thus the contribution of $T^n_{3,h}$ is far smaller than the bound we have to prove in 2.38. Next, for the contribution of $T^n_{2,h}$, we just write, since there is at most Ch^{-d} eigenvalues $\lambda_{j,h}$ and using the bound (2.35)

$$\begin{aligned}
T^n_{h,2}(x,y) &= \sum_{h^{-\alpha}<\lambda_{j,h}\le h^{-2}\delta_0} (1-h^2\lambda_{j,h})^n e_{j,h}(x)e_{j,h}(y) \\
\|T^n_{2,h}\|_{L^\infty\to L^\infty} &\le Ch^{-3d/2}(1-h^{2-\alpha})^n
\end{aligned} \tag{2.47}$$

Thus we get for some $C_\alpha > 0$,

$$\|T^n_{2,h}\|_{L^\infty\to L^\infty} \le C_\alpha e^{-\frac{nh^{2-\alpha}}{2}}, \quad \forall h, \quad \forall n \ge h^{-2+\alpha/2} \tag{2.48}$$

and thus this contribution is still neglectible for $h \in]0, h_0]$ for h_0 small. It remains to study the contribution of $T^n_{h,1}$. Let E_α be the (finite dimensional) subspace of $L^2(\rho)$ span by the eigenvectors $e_{j,h}, \lambda_{j,h} \le h^{-\alpha}$. By lemma 2.3, one has $dim(E_\alpha) \le Ch^{-d\alpha/2}$.

Lemma 2.4 *There exist $\alpha > 0$, $p > 2$ and C independent of h such that for all $u \in E_\alpha$, the following inequality holds true*

$$\|u\|^2_{L^p} \le Ch^{-2}((\mathcal{E}_{\Omega,h}(u) + h^2\|u\|^2_{L^2}) \tag{2.49}$$

Proof. Clearly, one has for $u = \sum_{\lambda_{1,h}\le\lambda_{j,h}\le h^{-\alpha}} a_j e_{j,h} \in E_\alpha$

$$\mathcal{E}_{\Omega,h}(u) + h^2\|u\|^2_{L^2} = \sum_{\lambda_{1,h}\le\lambda_{j,h}\le h^{-\alpha}} h^2(1+\lambda_{j,h})|a_j|^2$$

Take $u \in E_\alpha$ such that $h^{-2}((\mathcal{E}_{\Omega,h}(u)+h^2\|u\|^2_{L^2}) \le 1$. Then by 2.20, one has $h^{-2}\mathcal{E}_h(E(u)) \le C_0$. Let $\psi(t) \in C^\infty_0(\mathbb{R})$ equal to 1 near $t = 0$, and for $v(x) = \sum_{k\in\mathbb{Z}^d} c_k(v)e^{2i\pi kx/A}$, set

$$v = v_L + v_H, \quad v_L(x) = \sum_{k\in\mathbb{Z}^d} \psi(h|k|)c_k(v)e^{2i\pi kx/A} \tag{2.50}$$

Then $v = v_L + v_H$ is a decomposition of the extension $v = E(u)$ in low frequencies (v_L) and high frequencies (v_H). One has $v_L(x) = \int_{\mathbb{R}^d} h^{-d}\theta(\frac{x-y}{h})v(y)dy$, where θ is the function in the Schwartz space defined by $\hat\theta(2\pi z/A) = \psi(|z|)$. Hence, the map $v \mapsto v_L$ is bounded uniformly in h on all the space L^q for $1 \le q \le \infty$. Then, from 2.12 we get

$$\|v_L\|_{H^1(B)} \le C \tag{2.51}$$

Thus, with $u_L = v_L|_\Omega$ and $u_H = v_H|_\Omega$, we get $\|u_L\|_{H^1(\Omega)} \le C$ so by Sobolev for $p < \frac{2d}{d-2}$

$$\|u_L\|_{L^p} \le C \tag{2.52}$$

One the other hand, one has also by 2.20

$$h^{-2}\mathcal{E}_h(E(e_{j,h})) \le C_0(1+\lambda_{j,h}) \tag{2.53}$$

and this implies by 2.12

$$h^{-2}\|E(e_{j,h})_H\|_{L^2}^2 \le C_0(1+\lambda_{j,h}) \le C_0(1+h^{-\alpha}) \tag{2.54}$$

Thus for $\alpha \le 1$, we get $\|E(e_{j,h})_H\|_{L^2} \le Ch^{1/2}$. On the other hand, since $\|e_{j,h}\|_{L^\infty} \le Ch^{-d/2}$, using the definition of the low frequency cut-off we get ,

$$\|E(e_{j,h})_H\|_{L^\infty} \le \|E(e_{j,h})\|_{L^\infty} + \|E(e_{j,h})_L\|_{L^\infty} \le C\|E(e_{j,h})\|_{L^\infty} \le Ch^{-d/2}$$

By interpolation we can find some $p > 2$ such that

$$\|E(e_{j,h})_H\|_{L^p} \le C_0 h^{1/4} \tag{2.55}$$

Thus, one get for $u = \sum_{\lambda_{1,h}\le\lambda_{j,h}\le h^{-\alpha}} a_j e_{j,h} \in E_\alpha$ with $h^{-2}((\mathcal{E}_{\Omega,h}(u) + h^2\|u\|_{L^2}^2) \le 1$

$$\begin{aligned}\|u_H\|_{L^p} &\le \sum_{\lambda_{1,h}\le\lambda_{j,h}\le h^{-\alpha}} |a_j|\|E(e_{j,h})_H\|_{L^p} \\ &\le C_0 h^{1/4} dim(E_\alpha)^{1/2}\|u\|_{L^2} \le Ch^{1/4}h^{-d\alpha/4}\end{aligned} \tag{2.56}$$

Our lemma follows from 2.52 and 2.56 if one takes α small. Observe that here, the estimate on the number of eigenvalues (i.e the estimation of the dimension of E_α) is crucial. The proof of lemma 2.4 is complete. □

From lemma 2.4, using the interpolation inequality $\|u\|_{L^2}^2 \le \|u\|_{L^p}^{\frac{p}{p-1}}\|u\|_{L^1}^{\frac{p-2}{p-1}}$, we deduce the Nash inequality, with $1/D = 2 - 4/p > 0$

$$\|u\|_{L^2}^{2+1/D} \le Ch^{-2}((\mathcal{E}_{\Omega,h}(u) + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/D}, \quad \forall u \in E_\alpha \tag{2.57}$$

For $\lambda_{j,h} \le h^{-\alpha}$, one has $h^2\lambda_{j,h} \le 1$, and thus for any $u \in E_\alpha$, one gets $\mathcal{E}_{\Omega,h}(u) \le \|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2$, thus we get from 2.57

$$\|u\|_{L^2}^{2+1/D} \le Ch^{-2}((\|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2 + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/D}, \quad \forall u \in E_\alpha \tag{2.58}$$

From 2.46 and 2.48, and $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n + T_{h,3}^n$, we get that there exists C_2 such that

$$\|T_{1,h}^n\|_{L^\infty\to L^\infty} \le C_2, \quad \forall h, \quad \forall n \ge h^{-2+\alpha/2} \tag{2.59}$$

and thus since $T_{1,h}$ is self adjoint on L^2

$$\|T_{1,h}^n\|_{L^1\to L^1} \le C_2, \quad \forall h, \quad \forall n \ge h^{-2+\alpha/2} \tag{2.60}$$

Fix $p \simeq h^{-2+\alpha/2}$. Take $g \in L^2$ such that $\|g\|_{L^1} \le 1$ and consider the sequence $c_n, n \ge 0$

$$c_n = \|T_{1,h}^{n+p} g\|_{L^2}^2 \tag{2.61}$$

Then, $0 \le c_{n+1} \le c_n$ and from 2.58 and 2.60, we get

$$\begin{aligned}c_n^{1+\frac{1}{2D}} &\le Ch^{-2}(c_n - c_{n+1} + h^2 c_n)\|T_{1,h}^{n+p} g\|_{L^1}^{1/D} \\ &\le CC_2^{1/D}h^{-2}(c_n - c_{n+1} + h^2 c_n)\end{aligned} \tag{2.62}$$

From this inequality, we deduce that there exist $A \simeq CC_2 sup_{0\leq n \leq h^{-2}}(2+n)(1+h^2-(1-\frac{1}{n+2})^{2D})$ which depends only on C, C_2, D, such that for all $0 \leq n \leq h^{-2}$, one has $c_n \leq (\frac{Ah^{-2}}{1+n})^{2D}$, and thus there exist C_0 which depends only on C, C_2, D, such that for $N \simeq h^{-2}$, one has $c_N \leq C_0$. This implies

$$\|T_{1,h}^{N+p} g\|_{L^2} \leq C_0 \|g\|_{L^1} \tag{2.63}$$ T26

and thus taking adjoints

$$\|T_{1,h}^{N+p} g\|_{L^\infty} \leq C_0 \|g\|_{L^2} \tag{2.64}$$ T27

and so we get for any n and with $N+p \simeq h^{-2}$

$$\|T_{1,h}^{N+p+n} g\|_{L^\infty} \leq C_0(1-h^2\lambda_{1,h})^n \|g\|_{L^2} \tag{2.65}$$ T28

And thus for $n \geq h^{-2}$

$$\|T_{1,h}^{n}\|_{L^\infty \to L^\infty} \leq C_0 e^{-(n-h^{-2})h^2\lambda_{1,h}} = C_0 e^{\lambda_{1,h}} e^{-ngap}, \quad \forall h, \quad \forall n \geq h^{-2} \tag{2.66}$$ T29

This conclude the proof of theorem 1.1.

rem1 **Remark 2.5** *Observe that (2.35) is certainly true with a power of Λ instead of a power of h with $\lambda = 1 - h^2\Lambda$, but we have no proof for this; thats why we use for $T_{1,h}$ a Nash inequality.*

rem2 **Remark 2.6** *The above proof seems to apply for a more general choice of the elementary Markov kernel $h^{-d}\varphi(\frac{x-y}{h})$. Replace φ by a positive symmetric measure of total mass 1 with support in the unit ball, and let T_h be the Metropolis with this data. Assume that one is able to prove that for some $\delta_0 > 0$ one has $Spec(T_h) \subset [-1+\delta_0, 1]$ for all $h \leq h_0$, and that for some power M, one has for some $C, c > 0$*

$$T_h^M(x, dy) = \mu_h(x, dy) + Ch^{-d} 1_{|x-y|\leq ch} \rho(y) dy, \quad \mu_h(x, dy) \geq 0$$

Then there exist $\gamma < 1$ such that $\|\mu_h\|_{L^\infty} \leq \gamma$. Moreover, the right inequality in 2.20 and 2.35 are still valid for T_h^M. Also, the spectral gap of T_h^M is given by formula 1.4 with $T_h^M(x, dy)$ in place of $K_{h,\rho}(x, y)dy$, and therefore the left inequality in 1.6 holds true, and the right one is true , since if ρ is constant, for any $\theta \in C_0^\infty(\Omega)$, one has $u - T_h u \in O(h^2)$. We shall use these remark later in the study of the hard disc problem, in section 4.

3 A proof of theorem 1.2

sec3 In all this section, we suppose additionaly that Ω is stratified. For a given continuous density ρ, using (2.4) and an approximation of ρ in L^∞ by a sequence of smooth density ρ_k, one sees that the first assertion (1.18) of theorem 1.2 is a consequence of the second one (1.7). Assume now that ρ is smooth.

lem5 **Lemma 3.1** *Let $\theta \in C^\infty(\overline{\Omega})$ be such that $supp(\theta) \cap \Gamma_{sing} = \emptyset$ and $\partial_n \theta|_{\Gamma_{reg}} = 0$. Then*

$$Q_{h,\rho}(\theta) = h^2 L_\rho(\theta) + r, \quad \|r\|_{L^2} \in O(h^{5/2}) \tag{3.1}$$ 3.1

Proof. For $\theta \in C^\infty(\overline{\Omega})$ and $x \in \Omega$, we can use the Taylor formula to get

$$\begin{aligned} Q_{h,\rho}(\theta)(x) = \frac{1}{Vol(B_1)} \int_{A(x,h)} \min(1 + h\frac{\nabla\rho(x)}{\rho(x)}.z + O(h^2|z|^2), 1) \\ (-h\nabla\theta(x).z - \frac{h^2}{2}\sum_{i,j} z_i z_j \partial_{x_i}\partial_{x_j}\theta(x) + O(h^3|z|^3))dz \end{aligned} \tag{3.2}$$

with $A(x,h) = \{z \in \mathbb{R}^d,\ |z| < 1,\ x + hz \in \Omega\}$. As $A(x,h) = A^+(x,h) \cup A^-(x,h)$, with $A^\pm(x,h) = \{z \in A(x,h),\ \pm(\rho(x+hz) - \rho(x)) \geq 0\}$, it follows by an easy computation that

$$\begin{aligned} Q_{h,\rho}(\theta)(x) = &- \frac{h}{Vol(B_1)}\nabla\theta(x).\int_{A(x,h)} zdz - \frac{h^2}{2Vol(B_1)}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}\theta(x)\int_{A(x,h)} z_i z_j dz \\ &- \frac{h^2}{Vol(B_1)}\int_{A^-(x,h)} \frac{\nabla\rho(x)}{\rho(x)}.z\,\nabla\theta(x).zdz + r(x) = f_1(x) + f_2(x) + f_3(x) + r(x) \end{aligned} \tag{3.3}$$

with $\|r\|_{L^\infty(\Omega)} = O(h^3)$. Let $\chi = 1_{d(x,\partial\Omega)<2h}$, then for $j = 2, 3$

$$\|\chi f_j\|_{L^2(\Omega)} \leq \|\chi\|_{L^2(\Omega)}\|f_j\|_{L^\infty}(\Omega) = O(h^{5/2}) \tag{3.4}$$ 3.1bis

thanks to the support properties of χ. Moreover, for $x \in supp(1-\chi)$, $A(x,h) = \{|z| < 1\}$ and the change of variable $z \mapsto -z$ shows that $(1-\chi)f_2 = -(1-\chi)\frac{\alpha_d}{2}h^2\Delta\theta(x)$ thanks to 1.8. Hence,

$$f_2(x) = -\frac{\alpha_d}{2}h^2\Delta\theta(x) + r(x) \tag{3.5}$$

with $\|r\|_{L^2} = O(h^{5/2})$.

To compute f_3, we first observe that $|f_3(x)| \leq Ch^2|\nabla\rho(x)||\nabla\theta(x)|$. We thus get $\|1_{|\nabla\rho|\leq h^{1/2}}f_3\|_{L^\infty} \leq Ch^{5/2}\|\nabla\theta\|_{L^\infty}$. At a point x where $|\nabla\rho(x)| \geq h^{1/2}$, we may write $z = t\frac{\nabla\rho(x)}{|\nabla\rho(x)|} + z^\perp$, $\quad t = \frac{z.\nabla\rho(x)}{|\nabla\rho(x)|}$, $\quad z^\perp.\nabla\rho(x) = 0$. In these coordinates, one has $A^-(x,h) = \{(t, z^\perp), t|\nabla\rho(x)| + O(h(t^2 + |z^\perp|^2)) \leq 0\}$. From $|\nabla\rho(x)| \geq h^{1/2}$ we get that the symmetric difference R between $A^-(x,h)$ and $\{t \leq 0\}$ satisfies $meas(R) = O(h^{1/2})$ (the symmetric difference of two sets A, B is $A \cup B \setminus A \cap B$). Therefore

$$1_{|\nabla\rho|\geq h^{1/2}}(1-\chi)f_3(x) = -h^2 1_{|\nabla\rho|\geq h^{1/2}}(1-\chi)(x)\int_{\{|z|<1,\nabla\rho(x).z\leq 0\}} \frac{\nabla\rho(x)}{\rho(x)}.z\,\nabla\theta(x).zdz + r(x) \tag{3.6}$$

with $\|r\|_{L^\infty} = O(h^{5/2})$. Using the change of variable $z \mapsto z - 2z^\perp$ we get

$$1_{|\nabla\rho|\geq h^{1/2}}(1-\chi)f_3(x) = -h^2 1_{|\nabla\rho|\geq h^{1/2}}\frac{\alpha_d}{2}(1-\chi)(x)\frac{\nabla\rho(x)}{\rho(x)}.\nabla\theta(x) + r(x) \tag{3.7}$$

and therefore using 3.4 we get

$$f_3(x) = -h^2\frac{\alpha_d}{2}\frac{\nabla\rho(x)}{\rho(x)}.\nabla\theta(x) + r(x) \tag{3.8}$$

with $\|r\|_{L^2} = O(h^{5/2})$. It remains to show that $\|f_1\|_{L^2(\Omega)} = O(h^{5/2})$. Using the change of variable $z \mapsto -z$ we easily obtain $(1-\chi)f_1 = 0$. Hence, it suffices to show that $f_1'(x,h) =$

$\int_{A(x,h)} z.\nabla\theta(x)dz$ satisfies $\|f_1'\|_{L^\infty(\Omega)} = O(h)$. As Γ_{sing} is compact and $supp(\theta)\cap\Gamma_{sing} = \emptyset$, $dist(\Gamma_{sing}, supp(\theta)) > 0$, this is a local problem near any point x_0 of the regular part Γ_{reg} of the boundary . Let ϕ be a smooth diffeomorphism as in 2.8 so that locally near x_0, one has $\phi^{-1}(\Omega) = \{x_d > 0\}$. For x close to x_0 one has

$$A(x,h) = \{z \in \mathbb{R}^d,\ |z| < 1,\ (\phi^{-1}(x) + hD_x\phi^{-1}(z) + O(h^2))_d > 0\} \tag{3.9}$$

Set

$$A_1(x,h) = \{z \in \mathbb{R}^d,\ |z| < 1, (\phi^{-1}(x) + hD_x\phi^{-1}(z))_d > 0\} \tag{3.10}$$

then the symmetric difference R between $A(x,h)$ and $A_1(x,h)$ satisfies $meas(R) = O(h)$ uniformly in x close to x_0. This yields

$$f_1'(x,h) = \nabla\theta(x).v(x,h) + r(x), \quad v(x,h) = \int_{A_1(x,h)} zdz \tag{3.11}$$

with $\|r\|_{L^\infty} = O(h)$. Let $\nu(x)$ be the vector field defined by $\nu(x).z = (D_x\phi^{-1}(x)(z))_d$. Observe that $v(x,h)$ is collinear to $\nu(x)$, vanish for $dist(x,\partial\Omega) > Ch$ and that for $x \in \partial\Omega$, $\nu(x)$ is collinear to the unit normal to the boundary $\overrightarrow{n}(x)$. Since $\partial_n\theta|_{\Gamma_{reg}} = 0$, we thus get $\|f_1'\|_{L^\infty} = O(h)$. The proof of our lemma is complete. □

Let us recall that we denote $1 = \nu_0 < \nu_1 < \ldots < \nu_j < ...$ the eigenvalues of L_ρ and m_j the associated multiplicities. We introduce the bilinear form

$$a_\rho(u,v) = \frac{\alpha_d}{2}\langle\nabla u, \nabla v\rangle_{L^2(\rho)} + \langle u, v\rangle_{L^2(\rho)}. \tag{3.12}$$

It defines an Hilbertian structure on $H^1(\Omega)$ which is equivalent to the usual one. We denote $\|.\|_{H^1_\rho}$ the norm induced by a_ρ. For $j \in \mathbb{N}$ we denote $F_j = Ker(L_\rho - \nu_j)$, $F_{<N} = \oplus_{j<N}F_j$ and by $F_{\geq N} = \oplus_{j\geq N}F_j$ the orthogonal complement of $F_{<N}$ in H^1 for teh scalarr product a_ρ. Observe that since we assume here ρ smooth, by the classical theory of elliptic boundary problems, any function in F_j is smooth in Ω and smooth up to the regular part Γ_{reg} of the boundary. We also denote Π_j the orthogonal projection for a_ρ on F_j and

$$\mathcal{D}_N = \{\theta \in C^\infty(\overline{\Omega}), \theta = 0 \text{ near } \Gamma_{sing}, \partial_n\theta_{|\Gamma_{reg}} = 0, \langle\theta, v\rangle_{L^2(\rho)} = 0 \quad \forall v \in F_{<N}\} \tag{3.13}$$

where we use the convention $F_{<0} = \emptyset$. One has $\mathcal{D}_N \subset F_{\geq N}$, since for any $\theta \in \mathcal{D}_N$ and any $v \in F_j$ with $j < N$ one has by 1.14, $a_\rho(v,\theta) = \langle(L_\rho + 1)v, \theta\rangle_{L^2(\rho)} = \langle(\nu_j + 1)v, \theta\rangle_{L^2(\rho)} = 0$.

lem6 **Lemma 3.2** *For all $N \in \mathbb{N}$ and all $u \in F_{\geq N}$ there exists a sequence (u_k) in $\mathcal{D}_N$ converging to u in H^1.*

Proof. We proceed by induction. Let us first verify the property for $N = 0$, i.e that $\mathcal{D}_0$ is dense in H^1. Let $f \in H^1(\Omega)$ be orthogonal to $\mathcal{D}_0$ for a_ρ. Then, it is orthogonal to $C_0^\infty(\Omega)$ so that $(L_\rho + 1)f = 0$ in the sense of distributions. In particular $-\Delta f \in L^2(\Omega)$. Hence we can use the Green formula (1.14) to get for any $\theta \in \mathcal{D}_0$, since $a_\rho(f,\theta) = 0$,

$$\langle\partial_n f, \rho\theta\rangle_{H^{-1/2},H^{1/2}} = 0 \tag{3.14}$$

For any $\psi \in C_0^\infty(\Gamma_{reg})$, using smooth local coordinates we can find $\tilde{\psi}$ in $\mathcal{D}_0$ such that $\tilde{\psi}_{|\partial\Omega} = \psi$. Consequently,

$$\langle\partial_n f, \rho\psi\rangle_{H^{-1/2},H^{1/2}} = \langle\partial_n f, \rho\tilde{\psi}\rangle_{H^{-1/2},H^{1/2}} = 0 \tag{3.15}$$

Hence, $\partial_n f_{|\Gamma_{reg}} = 0$. This shows that $\partial_n f_{|\partial\Omega} \in H^{-1/2}$ is supported in Γ_{sing}. From 1.16 this implies $\partial_n f_{|\partial\Omega} = 0$. This shows that $f \in D(L_\rho)$. As the operator $L_\rho + 1$ is strictly positive this implies $f = 0$.

For $N \geq 1$ and $f \in F_{\geq N}$, we consider a familly (f_ϵ) in $\mathcal{D}_0$ such that $\|f - f_\epsilon\|_{H^1} \leq \epsilon$. Let Q be an open ball such that $\overline{Q} \subset \Omega$. We look for $h \in C_0^\infty(Q)$ such that $\tilde{f}_\epsilon = f_\epsilon + h$ satisfies the lemma. Let $\theta \in C_0^\infty(Q)$, with $\theta \geq 0$ and $\theta \neq 0$. We look for h under the form $h = \sum_{k=1}^J \beta_k \theta e_k$, where $(e_j)_{j\in\{1,\dots,J\}}$ denote the eigenfunctions of L_ρ such that $F_{<N} = span(e_j, j \in \{1,\dots,J\})$ and $\beta = (\beta_1,\dots,\beta_J) \in \mathbb{C}^J$. The condition $f_\epsilon + h \in \mathcal{D}_N$ reads $\langle h, e_j\rangle_{L^2(\rho)} = \alpha_j$ with $\alpha_j = -\langle f_\epsilon, e_j\rangle_{L^2(\rho)} = O(\epsilon)$. Denoting $\beta = (\beta_1,\dots,\beta_J)$ and $\langle u, v\rangle_{Q,\rho\theta} = \int_Q u(x)\overline{v}(x)\rho(x)\theta(x)dx$, this is equivalent to $M\beta = \alpha$ where M is the $J \times J$ matrix $M = (\langle e_j, e_k\rangle_{Q,\rho\theta})_{j,k=1,\dots,J}$.

We claim that $\langle , \rangle_{Q,\rho\theta}$ is definite positive on $F_{<N}$. If not, there will exist a non zero function $v \in F_{<N}$ such that $\int_Q |v(x)|^2\rho(x)\theta(x)dx = 0$. This implies that $v(x) = 0$ on the non void open set $\theta(x) > 0$. Since v satisfies $\Pi_{j<N}(L_\rho - \nu_j)v = 0$, the uniqueness theorem for second order elliptic operators implies $v(x) = 0$ for all $x \in \Omega$. As a consequence, the matrix M is invertible, so that $\beta = M^{-1}\alpha = O(\epsilon)$. Hence $\|h\|_{H^1} = O(\epsilon)$. The proof of our lemma is complete.

□

We are now in position to achieve the proof of Theorem 1.2. We first observe that if $\nu_h \in [0, M]$ and $\psi_h \in L^2(\rho)$ satisfy $\|\psi_h\|_{L^2} = 1$, $h^{-2}Q_h\psi_h = \nu_h\psi_h$, then thanks to Lemma 2.2 the family $(\psi_h)_{h\in]0,1]}$ is relatively compact in $L^2(\rho)$ so that we can suppose (extracting a subsequence h_k) that $\nu_h \to \nu$ and $\psi_h \to \psi$ in $L^2(\rho)$, $\|\psi\|_{L^2} = 1$, and moreover by Lemma 2.2, the limit ψ belongs to $H^1(\rho)$. Given $\theta \in \mathcal{D}_0$, it follows from self-ajointness of Q_h and Lemma 3.1 that

$$0 = \langle (h^{-2}Q_h - \nu_h)\psi_h, \theta\rangle_{L^2(\rho)} = \langle \psi_h, (L_\rho - \nu_h)\theta\rangle_{L^2(\rho)} + O(h^{1/2}) \tag{3.16}$$

Making $h \to 0$ we obtain $\langle \psi, (L_\rho - \nu)\theta\rangle_{L^2(\rho)} = 0$ for all $\theta \in \mathcal{D}_0$. It follows that $(L_\rho - \nu)\psi = 0$ in the distribution sense, and integrating by parts that $\partial_n\psi$ vanish on Γ_{reg}. Since $\psi \in H^1(\rho)$, we get as above using 1.16 that $\partial_n\psi = 0$, and it follows that $\psi \in D(L_\rho)$. This shows that ν is an eigenvalue of L_ρ, and thus (1.19) is satisfied. Moreover, by compactness in L^2 of the sequence ψ_h, one gets that for any $\epsilon > 0$ small enough, there exists $h_\epsilon > 0$ such that

$$\sharp Spec(h^{-2}Q_h) \cap [\nu_j - \epsilon, \nu_j + \epsilon] \leq m_j \tag{3.17}$$

for $h \in]0, h_\epsilon]$ with $h_\epsilon > 0$ small enough. It remains to show that there is equality in 3.17, and we shall proceed by induction on j.

Let $\epsilon > 0$ small be given such that for $0 \leq \nu_j \leq M+1$, the intervals $I_j^\epsilon = [\nu_j - \epsilon, \nu_j + \epsilon]$ are disjoint. Let $(\mu_j)_{j\geq 0}$ the increasing sequence of eigenvalues of $h^{-2}Q_h$, $\sigma_N = \sum_{j=1}^N m_j$ and $(e_k)_{k\geq 0}$ the eigenfunctions of L_ρ such that for all $k \in \{1 + \sigma_N, \dots, \sigma_{N+1}\}$, one has $(L_\rho - \nu_{N+1})e_k = 0$. As 0 is a simple eigenvalue of both L_ρ and Q_h, we have clearly $\nu_0 = \mu_0 = 0$ and $m_0 = 1 = \sharp Spec(h^{-2}Q_h) \cap [\nu_0 - \epsilon, \nu_0 + \epsilon]$.

Suppose that for all $n \leq N$, $m_n = \sharp Spec(h^{-2}Q_h) \cap [\nu_n - \epsilon, \nu_n + \epsilon]$. Then, one has by 1.19, for $h \leq h_\varepsilon$,

$$\mu_{1+\sigma_N} \geq \nu_{N+1} - \epsilon \tag{3.18}$$

By min-max principle, if G is a finite dimensional subspace of H^1 with $dim(G) = 1+\sigma_{N+1}$, one has

$$\mu_{\sigma_{N+1}} \leq \sup_{\psi\in G, \|\psi\|=1} \langle h^{-2}Q_h\psi, \psi\rangle_{L^2(\rho)} \tag{3.19}$$

Thanks to Lemma 3.2, for all $e_k, 0 \leq k \leq \sigma_{N+1}$ and all $\alpha > 0$, there exists $e_{k,\alpha} \in \mathcal{D}_0$ such that $\|e_k - e_{k,\alpha}\|_{H^1_\rho} \leq \alpha$. Let G_α be the vector space span by the $e_{k,\alpha}, 0 \leq k \leq \sigma_{N+1}$. For α small enough, one has $dim(G_\alpha) = 1 + \sigma_{N+1}$. From Lemma 3.1, one has

$$\langle h^{-2}Q_h e_{k,\alpha}, e_{k',\alpha}\rangle_{L^2(\rho)} = \langle L_\rho e_{k,\alpha}, e_{k',\alpha}\rangle_{L^2(\rho)} + O_\alpha(h^{1/2}) \tag{3.20}$$

Since $e_{k,\alpha} \in \mathcal{D}_0$, one has $\langle L_\rho e_{k,\alpha}, e_{k',\alpha}\rangle_{L^2(\rho)} = \frac{\alpha_d}{2}\langle \nabla e_{k,\alpha}, \nabla e_{k',\alpha}\rangle_{L^2_\rho}$, and $\langle \nabla e_{k,\alpha}, \nabla e_{k',\alpha}\rangle_{L^2_\rho} = \langle \nabla e_k, \nabla e_{k'}\rangle_{L^2_\rho} + O(\alpha)$. Therefore, for $\psi \in G_\alpha$, $\|\psi\| = 1$, we get

$$\langle h^{-2}Q_h\psi, \psi\rangle_{L^2(\rho)} \leq \nu_{N+1} + C\alpha + O_\alpha(h^{1/2}) \tag{3.21}$$

Taking $\alpha > 0$ small enough and $h < h_\alpha$ we obtain from 3.19, 3.21 $\mu_{\sigma_{N+1}} \leq \nu_{N+1} + \epsilon$. Combining this with 3.18 and 3.17, we get $m_{N+1} = \sharp Spec(h^{-2}Q_h) \cap [\nu_{N+1} - \epsilon, \nu_{N+1} + \epsilon]$. The proof of Theorem 1.2 is complete.

4 Application to random placement of non-overlapping balls

In this section, we suppose that Ω is a bounded Lipschitz stratified connected open subset of $\mathbb{R}^d$ with $d \geq 2$. Let $N \in \mathbb{N}, N \geq 2$ and $\epsilon > 0$ be given. Let $\mathcal{O}_{N,\epsilon}$ be the open bounded subset of $\mathbb{R}^{Nd}$

$$\mathcal{O}_{N,\epsilon} = \{x = (x_1, \dots, x_N) \in \Omega^N, \forall 1 \leq i < j \leq N, |x_i - x_j| > \epsilon\}$$

We introduce the kernel

$$K_h(x, dy) = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_1} \otimes \dots \otimes \delta_{x_{j-1}} \otimes h^{-d}\varphi(\frac{x_j - y_j}{h})dy_j \otimes \delta_{x_{j+1}} \otimes \dots \otimes \delta_{x_N} \tag{4.1}$$

and the associated Metropolis operator on $L^2(\mathcal{O}_{N,\epsilon})$

$$T_h(u)(x) = m_h(x)u(x) + \int_{O_{N,\epsilon}} u(y)K_h(x, dy) \tag{4.2}$$

with

$$m_h(x) = 1 - \int_{O_{N,\epsilon}} K_h(x, dy). \tag{4.3}$$

The operator T_h is Markov and self -adjoint on $L^2(\mathcal{O}_{N,\epsilon})$. The configuration space $\mathcal{O}_{N,\epsilon}$ is the set of N disjoints closed balls of radius $\epsilon/2$ in $\mathbb{R}^d$, with centers at the $x_j \in \Omega$. The topology of this set, and the geometry of its boundary is in general hard to understand (**references a trouver**), but since $d \geq 2$, $\mathcal{O}_{N,\epsilon}$ is clearly non void and connected for a given N if ϵ is small enough. The metropolis kernel T_h is associated to the following algorithm: at each step, we choose uniformly at random a ball, and we move it center uniformly at random in $\mathbb{R}^d$ in a ball of radius h. If the new configuration is in $\mathcal{O}_{N,\epsilon}$, the change is made. Otherwise, the configuration is kept as it started.

In order to study the random walk associated to T_h in in Proposition 4.1 we prove that the open set $\mathcal{O}_{N,\epsilon}$ is Lipschitz stratified for $\epsilon > 0$ small enough, and in lemma 4.5 we prove that the kernel of the iterated operator T_h^M (with M large, but independent of h) admits a suitable lower bound, so that we will be able to use our remark 2.6.

We define Γ_{reg} and Γ_{sing} the set of regular and singular points of $\partial\mathcal{O}_{N,\epsilon}$ as follows. Let us denote $\mathbb{N}_N = \{1, \dots, N\}$. For $x \in \overline{\mathcal{O}}_{N,\epsilon}$ we set

$$\begin{aligned} &R(x) = \{i \in \mathbb{N}_N,\ x_i \in \partial\Omega\} \\ &S(x) = \{\tau = (\tau_1, \tau_2) \in \mathbb{N}_N,\ \tau_1 < \tau_2 \text{ and } |x_{\tau_1} - x_{\tau_2}| = \epsilon\} \\ &r(x) = \sharp R(x),\ s(x) = \sharp S(x) \end{aligned} \tag{4.4}$$

The functions r and s are lower semi-continuous and any $x \in \overline{\mathcal{O}}_{N,\epsilon}$ belongs to $\partial\mathcal{O}_{N,\epsilon}$ iff $r(x) + s(x) \geq 1$. We define

$$\begin{aligned} \Gamma_{reg} = \{x \in \overline{\mathcal{O}}_{N,\epsilon}, s(x) = 1 \text{ and } r(x) = 0\} \\ \cup \{x \in \overline{\mathcal{O}}_{N,\epsilon},\ s(x) = 0, R(x) = \{j_0\} \text{ and } x_{j_0} \in \partial\Omega_{reg}\} \end{aligned} \tag{4.5}$$

and $\Gamma_{sing} = \partial\mathcal{O}_{N,\epsilon} \setminus \Gamma_{reg}$. Then, Γ_{sing} is clearly close, and the Γ_{reg} is the reunion of smooth disjoint hypersurfaces in $\mathbb{R}^{Nd}$.

prop7 **Proposition 4.1** *For $\epsilon > 0$ small enough, the set $\mathcal{O}_{N,\epsilon}$ is connected, Lipschitz and stratified.*

Proof. For $\nu \in S^{p-1}$, $p \geq 1$ and $\delta \in]0, 1[$ we denote

$$\Gamma_{\pm}(\nu, \delta) = \{\xi \in \mathbb{R}^p,\ \pm\langle\xi, \nu\rangle > (1 - \delta)|\xi|,\ |\langle\xi, \nu\rangle| < \delta\}. \tag{4.6}$$

We remark also that an open set $\mathcal{O} \subset \mathbb{R}^p$ is Lipschitz if it satisfies the cone property: $\forall a \in \partial\mathcal{O}, \exists\delta > 0, \exists\nu_a \in S^{p-1}, \forall b \in B(a, \delta) \cap \partial\mathcal{O}$ we have

$$b + \Gamma_+(\nu_a, \delta) \subset \mathcal{O} \text{ and } b + \Gamma_-(\nu_a, \delta) \subset \mathbb{R}^p \setminus \overline{\mathcal{O}}. \tag{4.7}$$

Let $\overline{x} \in \partial\mathcal{O}_{N,\epsilon}$. The equivalence relation $i \simeq j$ iff $\overline{x}_i$ and $\overline{x}_j$ can be connected by a path lying in the union of the closed balls, give us a partition $\{1, \dots, N\} = \cup_{k=1}^r F_k$ such that

$$\begin{aligned} &\forall k \neq l, \forall i \in F_k, \forall j \in F_l, |\overline{x}_i - \overline{x}_j| > \epsilon \\ &\forall k, \forall i \neq j \in F_k, \exists(n_l) \in F_k, 1 \leq l \leq m,\ n_1 = i, n_m = j,\ |\overline{x}_{n_l} - \overline{x}_{n_{l+1}}| = \epsilon. \end{aligned} \tag{4.8}$$

Observe that in the case where $\sharp F_k = 1$ the second condition is empty. We look for $\nu \in S^{Nd-1}$ such that the cone property at $\overline{x}$ holds with ν. We construct the coordinates of ν according to the partition $(F_k)_k$. Let $k \in \{1, \dots, r\}$.

Suppose that $F_k = \{j_k\}$ for some $j_k \in \{1, \dots, N\}$. If $\overline{x}_{j_k} \in \Omega$, we set $\xi_{j_k} = 0$. If $\overline{x}_{j_k} \in \partial\Omega$, thanks to the cone property satisfied by Ω we can find $\xi_{j_k} \in S^{d-1}$ and $\delta_k > 0$ such that

$$\overline{x}_{j_k} + \Gamma_+(\xi_{j_k}, \delta_k) \subset \Omega \text{ and } \overline{x}_{j_k} + \Gamma_-(\xi_{j_k}, \delta_k) \subset \mathbb{R}^d \setminus \overline{\Omega}. \tag{4.9}$$

Suppose that $\sharp F_k \geq 2$. If $\overline{x}_j \in \Omega$ for all $j \in F_k$ we fix $y_k \in \mathbb{R}^d$ and for all $j \in F_k$ we set $\xi_j = \overline{x}_j - y_k$. Hence $x_j(t) = \overline{x}_j + t\xi_j$ is obtained from $\overline{x}_j$ by homothety centered in y_k, with ratio $1+t$. Consequently, for all $i, j \in F_k$ and t close to 0, we have $|x_i(t) - x_j(t)| > \epsilon$ for $t > 0$ and there exists at least two distinct indices $i, j \in F_k$ such that $|x_i(t) - x_j(t)| < \epsilon$ for $t < 0$.

If there exists $i_0 \in F_k$ such that $\overline{x}_{i_0} \in \partial\Omega$, the cone property provides us $\eta_{i_0} \in S^{d-1}$ and $\delta_{i_0} > 0$. Moreover, taking $\epsilon > 0$ sufficiently small, we can suppose that all the $\overline{x}_i, i \in F_k$ are close to $\overline{x}_{i_0}$, since $|\overline{x}_i - \overline{x}_{i_0}| \leq \epsilon(N-1)$. Thus, for all $i \in F_k$, we have $b + \Gamma_+(\eta_{i_0}, \delta_{i_0}) \subset \Omega$ for all $b \in \overline{\Omega}, |b - \overline{x}_i| \leq \delta_{i_0}$. Let $y_{i_0}(z) = \overline{x}_{i_0} - z$
eta_{i_0} and for $i \in F_k$, $\xi_i'(z) = \overline{x}_i - y_{i_0}(z)$. Then, for $z > 0$ large enough and $\alpha > 0$ sufficiently small, we have $\xi_i = \alpha\xi_i'(z) \in \Gamma(\eta_{i_0}, \delta_{i_0})$. In particular, for $i \in F_k$ and $t > 0$ close to 0, $\overline{x}_i + t\xi_i$ belongs to Ω, while for $t < 0$, one has $\overline{x}_{i_0} + t\xi_{i_0} \notin \overline{\Omega}$. Moreover, the same argument as above shows that for $i \neq j$, the functions $t \mapsto |\overline{x}_i - \overline{x}_j + t(\xi_i - \xi_j)|$ are strictly increasing near $t = 0$.

Observe that $\xi = (\xi_1, \ldots, \xi_N) \neq 0$, since if $\xi_i = 0$ for all i, then $\sharp F_k = 1$ and $\overline{x}_{j_k} \in \Omega$ for all k which contradicts $\overline{x} \in \partial\mathcal{O}_{N,\epsilon}$. Finally, we take $\nu = \frac{\xi}{|\xi|}$. Then for $\delta > 0$ small enough, the cone property is satisfied at $\overline{x}$ with (ν, δ). Thus $\mathcal{O}_{N,\epsilon}$ is Lipschitz.

Let us show that $\mathcal{O}_{N,\epsilon}$ is connected for ϵ small enough. We define for $j \in \mathbb{N}_N$ the two applications π_j from $\mathbb{R}^{Nd}$ to $\mathbb{R}^{Nd}$ and σ_j from $\mathbb{R}^d$ to $\mathbb{R}^{Nd}$ by

$$\begin{aligned} \pi_j(x_1, ..., x_j, ..., x_N) &= (x_1, ..., 0, ..., x_N) \\ \sigma_j(y) &= (0, ..., y, ..., 0) \end{aligned} \tag{4.10}$$ g0

so that $x = \pi_j(x) + \sigma_j(x_j)$. For $F \subset \mathbb{N}_N$ we define $\sigma_F : \Pi_{j\in F}\mathbb{R}^d \to \mathbb{R}^{Nd}$ by $\sigma_F(x) = \sum_{j\in F} \sigma_j(x_j)$. We have the following geometric lemma.

lem7ter **Lemma 4.2** *There exists $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0]$ and all $n \in \mathbb{N}_N$, there exists a finite covering $(U_l)_l$ of $\mathcal{O}_{N,\epsilon}$ such that for all l there exists a subset $F_n \subset \mathbb{N}_N$ with $\sharp F_n = n$, there exists $\nu \in S^{nd-1}$ such that*

$$\forall x \in U_l \cap \mathcal{O}_{N,\epsilon},\ x + \sigma_{F_n}(\Gamma_+(\nu, \delta_0)) \subset \mathcal{O}_{N,\epsilon} \tag{4.11}$$ s100

Moreover, there exists $c_0 > 0$ such that for all $k, l \in F_n$ with $k \neq l$ and for $t \in [0, \delta_0]$, we have

$$\forall x \in U_l \cap \mathcal{O}_{N,\epsilon},\ |x_k + t\nu_k - x_l - t\nu_l|^2 \geq \epsilon^2 + c_0\epsilon t \tag{4.12}$$ s10

Proof. This lemma means that we can select an arbitrary number of balls n, and that moving only these balls by a vector in $\Gamma_+(\nu, \delta_0)$ while keeping the other balls fixed, results in an admissible configuration. We shall proceed by induction on $N \geq 1$. For $N = 1$, this is true since Ω is Lipschitz. Let $N \geq 2$ and suppose that the property is true until rank $N - 1$ and let $\tilde{\epsilon}_0$ be the corresponding parameter. For $\epsilon \in]0, \tilde{\epsilon}_0[$ and $\beta > 0$, we have the partition

$$\mathcal{O}_{N,\epsilon} = \mathcal{U}_{N,\epsilon,\beta} \cup \mathcal{V}_{N,\epsilon,\beta} \tag{4.13}$$

with $\mathcal{U}_{N,\epsilon,\beta} = \{x \in \mathcal{O}_{N,\epsilon}, \sup_{i\neq j} |x_i - x_j| < \epsilon + \beta\}$. Using the induction hypothesis, it is easy to see that for any $\beta > 0$, there exists $\tilde{\epsilon}_1(\beta) < \tilde{\epsilon}_0$ and $\tilde{\delta}_0(\beta) > 0$ such that the conclusion of the Lemma holds true on $\mathcal{V}_{N,\epsilon,\beta}$, $\forall \epsilon \in]0, \tilde{\epsilon}_1(\beta)]$. Hence it remains to find a suitable covering of $\mathcal{U}_{N,\epsilon,\beta}$. For any $\alpha>0$, choosing β and $\tilde{\epsilon}_1$ small enough we can suppose that

$$\mathcal{U}_{N,\epsilon,\beta}\subset \cup_{y^0\in F_\alpha} B(y^0,\alpha)^N \tag{4.14}$$

for some finite set $F_\alpha\subset\Omega$. Moreover, since Ω is Lipschitz, we can suppose that α is sufficiently small so that

$$\exists\delta_0'>0,\ \forall y^0\in F_\alpha,\ \exists v(y^0)\in S^{d-1},\ \forall y\in B(y^0,\alpha),\ y+\Gamma_+(v(y^0),2\delta_0')\subset\Omega. \tag{4.15}$$

Moreover, we can suppose that δ_0' is sufficiently small so that the following holds true:

$$\begin{aligned}&\forall v,u_1,\dots,u_N\in S^d,\ \text{s.t. } \forall j, \langle v,u_j\rangle=0,\ \exists v'\in\Gamma_+(v,2\delta_0'),\\ &\forall\xi\in\Gamma_+(v',\delta_0'),\forall j,\ \langle\xi,u_j\rangle\neq 0.\end{aligned} \tag{4.16}$$

Indeed, the set $A=\{v\in S^d,\ \langle v,u_j\rangle=0,\forall j\}$ is contained in a finite union of equators and thus (4.16) is obvious by taking v' close to v in the complementary.

Condition (4.15) gives us a critical value for $\tilde{\epsilon}_1$ and we now suppose that $\epsilon\in]0,\tilde{\epsilon}_1]$. By compactness, it remains to show that $y_0\in F_\alpha$ being fixed the following property holds true:

$$\begin{aligned}&\forall x^0\in B(y^0,\alpha)^N\cap\mathcal{O}_{N,\epsilon},\forall n\in\mathbb{N}_N,\exists F_n\subset\mathbb{N}_N \text{ s.t. } \sharp F_n=n,\ \exists\nu\in S^{nd-1},\ \exists r>0,\ \text{s.t.}\\ &\forall x\in B(x^0,r)\cap\mathcal{O}_{N,\epsilon},\ \forall\xi\in\Gamma_+(\nu,\delta_0'),\forall k\neq j,\ |(x+\sigma_F(\xi))_j-(x+\sigma_F(\xi))_k|>\epsilon\end{aligned} \tag{4.17}$$

Let $x^0\in B(y^0,\alpha)^N$ and $n\in\mathbb{N}_N$ being fixed. We construct F_n and $\nu\in S^{nd-1}$ by induction on n. We look for ν under the form $\nu=\lambda(v,(1-1/N)v,\dots,(1-(n-1)/N)v)$, where $v\in S^{d-1}$ and λ is a normalizing constant. We claim that we can find $F_n=\{j_1,\dots j_n\}$ and $v\in S^{d-1}$ such that

$$\langle v,x^0_{j_1}\rangle>\dots>\langle v,x^0_{j_n}\rangle>\langle v,x^0_s\rangle,\ \forall s\notin F_n \tag{4.18}$$

If $n=1$ we denote F' the set of index i such that the map $s\in\mathbb{N}_N\mapsto\langle v(y^0),x^0_s\rangle$ has a maximum in $s=i$. If $\sharp F'=1$ then we can take $(F_1,v)=(F',v(y^0))$. If $\sharp F'\geq 2$, thanks to (4.17) we can find v close to $v(y^0)$ such that $s\mapsto\langle v,x^0_s\rangle$ has a unique maximum for some $s=j_1$ and we set $F_1=\{j_1\}$.

Suppose now that $n\geq 1$ and that (F_n,v) satisfies (4.18). Let F'' be the set of index $i\notin F_n$ such that the map $s\notin F_n\mapsto\langle v,x^0_s\rangle$ has a maximum in $s=i$. If $\sharp F''=1$ then $(F_{n+1},v)=(F_n\cup F'',v)$ satisfies the expected property. If $\sharp F''\geq 2$ we can find v'' close to v so that we still have

$$\langle v'',x^0_{j_1}\rangle>\dots>\langle v'',x^0_{j_n}\rangle>\langle v'',x^0_s\rangle,\ \forall s\notin F_n \tag{4.19}$$

and additionally $\langle x_k,v''\rangle\neq\langle x_l,v''\rangle$ for all $k,l\in F''$ with $k\neq l$, and $\langle x_k,v''\rangle>\langle x_s,v''\rangle$ for all $k\in F'', s\notin F_n\cup F''$. This permits to find easily $j_{n+1}\in F''$ such that setting $F_{n+1}=F_n\cup\{j_{n+1}\}$, (4.18) holds true at rank $n+1$.

We turn back to the proof of (4.17). Let $x\in B(x^0,r)$ and $\xi=(\xi_{j_1},\dots,\xi_{j_n})\in\Gamma_+(\nu,\delta_0')$ with δ_0' given by the above construction and $r>0$ to be chosen small enough. We denote $\nu=(\nu_{j_1},\dots,\nu_{j_n})$ so that $\nu_{j_k}=\lambda(1-(k-1)/N)v$. The fact that $x+\sigma_{F_n}(\xi)\in\Omega^N$ is a straightforward consequence of (4.15) and it remains to show that the distance between two different balls remains bounded from below by ϵ. Let $j,k\in\mathbb{N}_N$ with $j\neq k$. If $j,k\notin F_n$, there is nothing to prove.

In the case where $j,k \in F_n$, we can write $j = j_p$ and $k = j_q$ with $p < q$ and we have

$$
\begin{aligned}
|(x+\sigma_F(\xi))_j - (x+\sigma_F(\xi))_k|^2 &\geq \epsilon^2 + 2\langle x_{j_p} - x_{j_q}, \xi_{j_p} - \xi_{j_q}\rangle \\
&\geq \epsilon^2 + 2|\xi|\langle x^0_{j_p} - x^0_{j_q}, \nu_{j_p} - \nu_{j_q}\rangle - 4r(|\xi_{j_p}| + |\xi_{j_q}|) \\
&\qquad - 2|\xi||x^0_{j_p} - x^0_{j_q}|(|\frac{\xi_{j_p}}{|\xi|} - \nu_{j_p}| + |\frac{\xi_{j_q}}{|\xi|} - \nu_{j_q}|) \\
&\geq \epsilon^2 + 2\frac{\lambda}{N}(q-p)|\xi|\langle v, x^0_{j_p} - x^0_{j_q}\rangle - 8r|\xi| - 4|\xi|\delta_0'|x^0_{j_p} - x^0_{j_q}|
\end{aligned}
\tag{4.20}
$$

Thanks to (4.18), $\langle v, x^0_{j_p} - x^0_{j_q}\rangle > c_1|x^0_{j_p} - x^0_{j_q}|$, for some $c_1 > 0$ independant on ϵ. Hence, taking $\delta_0' \in]0, \lambda c_1/(8N)]$ and $r \in]0, \lambda\epsilon/(16N)]$, we get

$$
|(x+\sigma_F(\xi))_j - (x+\sigma_F(\xi))_k|^2 > \epsilon^2 + c_1\frac{\lambda\epsilon}{N}|\xi| \tag{4.21}
$$

so that (4.11) is satisfied. Moreover, as λ is of order $n^{-1/2}$, (4.12) holds true with $c_0 = \frac{c_1}{N\sqrt{n}}$.

In the case $j \in F_n$ and $k \notin F_n$, similar computation leads to the same result. Taking $\delta_0 = \min(\tilde{(}\delta_0(\beta), \delta_0')$, the proof of the lemma is complete. □

sublemma1 **Lemma 4.3** *There exists $\epsilon_1 > 0$ and $m \geq 1$ such that $\forall \epsilon \in]0, \epsilon_1]$, $\forall x, y \in \mathcal{O}_{N,m\epsilon}$ satisfiying $\inf_{j,k}|x_j - y_k| > m\epsilon$, there exists a continuous path $\gamma : [0,1] \to \mathcal{O}_{N,\epsilon}$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proof. For $z' = (z_2, \dots, z_N) \in \Omega^{N-1}$ and $k \geq 1$ we denote $\tilde{\Omega}_k(z') = \Omega \setminus \cup_{j=2}^N \overline{B}(z_j, k\epsilon)$. We claim that there exists $\epsilon_1 > 0$ such that for $\epsilon \in]0, \epsilon_1]$ and $m \geq 1$ large enough, the following property holds true:

$$
\forall x' \in \mathcal{O}_{N-1,m\epsilon}, \forall u, v \in \tilde{\Omega}_m(x'), \exists \gamma \in C([0,1], \tilde{\Omega}_1(x')), \text{ s.t. } \gamma(0) = u, \gamma(1) = v \tag{4.22}
$$ s15

Indeed, Ω being connected, for any $u, v \in \tilde{\Omega}_m(x')$ there exists a continous path $\gamma : [0,1] \to \Omega$ from u to v. Moreover, using the fact that $\partial\Omega$ is Lipschitz and taking m sufficiently large, we can modify the path γ in a path $\tilde{\gamma}$ that avoids the balls $\overline{B}(x_j, \epsilon)$ and remains in Ω.

Now, let $x, y \in \mathcal{O}_{N,m\epsilon}$ with $\inf_{j,k}|x_j - y_k| > m\epsilon$. Thanks to (4.22), we can find a continous path from x to $(y_1, x_2, \dots, x_N)$ with values in $\mathcal{O}_{N,\epsilon}$. As $|y_1 - x_j| > \lambda$ for all j, we can apply (4.22) with $z' = (y_1, x_3, \dots, x_N)$ so that we can find a continuous path in $\mathcal{O}_{N,\epsilon}$ joining $(y_1, x_2, \dots, x_N)$ and $(y_1, y_2, x_3, \dots, x_N)$. Iterating this process we obtain a continous path from x to y, with values in $\mathcal{O}_{N,\epsilon}$. □

We are now in position to prove that $\mathcal{O}_{N,\epsilon}$ is connected for ϵ small enough. Let $\epsilon_0, \delta_0 > 0$ be given by Lemma 4.2 and m, ϵ_1 be given by Lemma 4.3. We can also decrease ϵ_1 so that $\forall \epsilon \in]0, \epsilon_1[$, $\forall x \in \mathcal{O}_{N,\epsilon}$, $\exists y \in \mathcal{O}_{N,m\epsilon}$ s.t. $\inf_{i,j}|x_i - y_j| > 2m\epsilon$. Let $\epsilon \leq \min(\epsilon_1, \epsilon_0, c_0\delta_0/(m^2 - 1))$ with c_0 given by Lemma 4.2.

Thanks to Lemma 4.3, it suffices to show that for any $x \in \mathcal{O}_{N,\epsilon}$ there exists a continuous path $\gamma : [0,1] \to \mathcal{O}_{N,\epsilon}$ such that $\gamma(0) = x$ and $\gamma(1) \in \mathcal{O}_{N,m\epsilon}$.

Let $x \in \mathcal{O}_{N,\epsilon}$ be fixed. Thanks to Lemma 4.2, there exists $\nu \in S^{Nd-1}$ such that the segment $x + [0, \delta_0]\nu$ is contained in Ω^N and moreover for any $k \neq l$ and any $t \in [0, \delta_0]$,

$$
|x_k + t\nu_k - x_l - t\nu_l|^2 \geq \epsilon^2 + c_0\epsilon t. \tag{4.23}
$$

Hence, the path $t \in [0,1] \mapsto \gamma(t) = x + t\delta_0\nu$ has the required properties. This achieves to prove that $\mathcal{O}_{N,\epsilon}$ is connected.

Let us now prove that $\mathcal{O}_{N,\epsilon}$ is stratified. Let $u \in H^{-1/2}(\mathcal{O}_{N,\epsilon})$ be supported in Γ_{reg}. We have to show that u is identically zero. This is a local problem and we can suppose that u is supported in a small open set $U \subset \mathbb{R}^{Nd}$ such that $\tilde{U} := U \cap \mathcal{O}_{N,\epsilon} = \{(x_1+\varphi(x'), x'), x_1 \in]0,\alpha[, x' \in V\}$, $\delta U := U \cap \partial\mathcal{O}_{N,\epsilon} = \{(\varphi(x'), x'), x' \in V\}$, where $\varphi : V \subset \mathbb{R}^{Nd-1} \to \mathbb{R}$ is a Lipschitz function. Denote

$$\begin{aligned} \kappa :]0,\alpha[\times V &\to \tilde{U}, & k : V &\to \delta U \\ (x_1, x') &\mapsto (x_1 + \varphi(x'), x') & x' &\mapsto (\varphi(x'), x') \end{aligned} \tag{4.24}$$

and for $\chi \in C_0^\infty(V)$, let $\phi_\chi(u)$ be defined by

$$\langle \phi_\chi(u), f\rangle = \langle u, (\chi f) \circ k^{-1}\rangle_{H^{-1/2}(\partial O_{N,\epsilon}), H^{1/2}(\partial O_{N,\epsilon})} \tag{4.25}$$

for any $f \in H^{1/2}(\mathbb{R}^{Nd-1})$. Then $\phi_\chi(u) \in H^{-1/2}(\mathbb{R}^{Nd-1})$ and $supp(\phi_\chi(u)) \subset k^{-1}(supp(u))$. Moreover, the distribution $\tilde{v} = \delta_{x_1=0} \otimes \phi_\chi(u)$ belongs to $H^{-1}(\mathbb{R}^{Nd})$ and $supp(\tilde{v}) \subset \kappa^{-1}(supp(u))$.

Let $\overline{x} \in \delta U \cap supp(u)$ and denote $\mathcal{D}_{N,\epsilon} = \{x \in (\mathbb{R}^d)^N,\ |x_i - x_j| > \epsilon,\ \forall 1 \le i < j \le N\}$. Then, either $r(\overline{x}) + s(\overline{x}) \ge 2$, either $\overline{x} \in \mathcal{D}_{N,\epsilon}$, $R(\overline{x}) = \{j_0\}$ (say j_0=1) and $\overline{x}_{j_0} \in \partial\Omega_{sing}$. Suppose that we are in the second case and let χ be a cut-off function supported near $\overline{x}$ such that $supp(\chi) \subset \mathbb{R}^d \times \Omega^{N-1} \cap \mathcal{D}_{N,\epsilon}$. Then, for any $\psi \in C_0^\infty(\overline{\Omega}^{N-1})$ the linear form u_ψ defined on $H^{1/2}(\partial\Omega)$ by

$$\langle u_\psi, f\rangle = \langle \chi u, f(x_1)\psi(x_2, \dots, x_N)\rangle_{H^{-1/2}(\delta U), H^{1/2}(\delta U)} \tag{4.26}$$

is continuous and supported in $\partial\Omega_{sing}$. As $\partial\Omega$ is stratified, it follows that u_ψ is equal to zero for all ψ and hence, $\chi u = 0$. Therefore, we can suppose that u is supported in the set $\{r(x) + s(x) \ge 2\}$. For $n \in \mathbb{N}$, $n \ge 2$, let us introduce the following property

$$(\mathcal{P}_n): \text{ for any } \overline{x} \in \delta U \text{ s.t. } r(\overline{x}) + s(\overline{x}) = n, \text{ we have } u = 0 \text{ near } \overline{x}. \tag{4.27}$$

We prove this property by induction on n. We first assume $n = 2$ and suppose that $r(\overline{x}) = s(\overline{x}) = 1$ (the cases $r = 2, s = 0$ and $r = 0, s = 2$ are similar and left to the reader). By lower semicontinuity of the functions r and s, for any $x \in supp(u)$ close to $\overline{x}$ we have also $r(x) = s(x) = 1$ and hence $R(x) = R(\overline{x})$ and $S(x) = S(\overline{x})$. Hence, we can suppose without loosing generality, that u is supported in $G = \partial\Omega \times \Omega^{N-1} \cap \{|x_i - x_2| = \epsilon\}$ for some $i \in \{1, 3, \dots, N\}$. Denoting $x_i = (x_{i,1}, \dots, x_{i,d})$ and using the fact that $\partial\mathcal{D}_{N,\epsilon}$ is invariant under any transformation of the form $x \mapsto (\rho(x_1), \dots, \rho(x_N))$ where ρ is an affine isometry of $\mathbb{R}^d$, there exists a linear map L on $\mathbb{R}^{Nd}$ such that $L(G)$ is given by two equations

$$\begin{aligned} x_{1,1} &= \alpha(x_1') \\ x_{2,1} &= \beta(x_2', x_i). \end{aligned} \tag{4.28}$$

with α Lipschitz and β smooth and where $x_j' = (x_{j,2}, \dots, x_{j,d})$. Hence, $\nu(x) = (x_{1,1} - \alpha(x_1'), x_{2,1} - \beta(x_2', x_i), x_1', x_2', x_3, \dots, x_N)$ defines a local homeomorphism of $\mathbb{R}^{Nd}$ such that $\nu \circ L(G) \subset \{0\}^2 \times \mathbb{R}^{Nd-2}$. Consequently, $\tilde{w} \in H^{-1}(\mathbb{R}^{Nd})$ defined by

$$\langle \tilde{w}, f\rangle = \langle \tilde{v}, f \circ \nu \circ L \circ \kappa\rangle \tag{4.29}$$

satisfies $supp(\tilde{w}) \subset \{0\}^2 \times \mathbb{R}^{Nd-2}$. Therefore, $\tilde{w}$ vanishes identically and hence u is null near $\overline{x}$.

Suppose now that $(\mathcal{P}_k)$ holds for $k \leq n$ and let $\overline{x}$ be such that $r(\overline{x})+s(\overline{x}) = n+1$. The lower semicontinuity of r, s and the induction hypothesis show that for any $x \in supp(u)$ close enough to $\overline{x}$, we have $r(x) + s(x) = n + 1$ and hence $R(x) = R(\overline{x})$, $S(x) = S(\overline{x})$. Suppose that $r(\overline{x}) = 0$, then $s(\overline{x}) = n + 1$ and near $\overline{x}$, $supp(u)$ is contained in $G = \Omega^N \cap (\cap_{\tau \in R(\overline{x})} \{|x_{\tau_1} - x_{\tau_2}| = \epsilon\}$. In particular, there exists $\sigma, \tau \in R(\overline{x})$ such that $\sigma \neq \tau$. As in the case $n = 2$, we can suppose that near $\overline{x}$, the set $\{r(x)+s(x) = n+1\}$ is contained in $\{x_{\tau_1,1} = \alpha(x'_{\tau_1}, x_{\tau_2}), x_{\sigma_1,1} = \alpha(x'_{\sigma_1}, x_{\sigma_2})\}$ for some Lipschitz functions α, β (here we forget some information). Hence, we can construct as precedently an homeomorphism ν on $\mathbb{R}^{Nd}$ such that $\nu(G) \subset \{0\}^2 \times \mathbb{R}^{Nd-2}$ and the same proof as for $n = 2$ still works. The cases $s = 0, r = n + 1$ and $s \geq 1, r \geq 1$ are similar and left to the reader.

The proof of proposition prop7 4.1 is complete. □

remk5 **Remark 4.4** *Observe that in the above lemma, the smallness condition on ϵ is $N\epsilon < c$ where $c > 0$ depends only on Ω. The condition $N\epsilon^d < c$, which say that the density of the balls is small enough, does not implies that the set $\mathcal{O}_{N,\epsilon}$ has Lipschitz regularity. As an example, if $\Omega =]0, 1[^2$ is the unit square in the plane, then $x = (x_1, ..., x_N)$, $x_j = ((j - 1)\epsilon, 0)$, $j = 1, ..., N$, with $\epsilon = \frac{1}{N-1}$ is a configuration point in the boundary $\partial\mathcal{O}_{N,\epsilon}$. However, $\partial\mathcal{O}_{N,\epsilon}$ is not Lipschitz at x: otherwise, there will exist $\nu_j = (a_j, b_j)$ such that $(x_1 + t\nu_1, ..., x_N + t\nu_N) \in \mathcal{O}_{N,\epsilon}$ for $t > 0$ small enough, and this implies $a_1 > 0, a_{j+1} > a_j$ and $a_N < 0$ which is impossible.*

For $k \in \mathbb{N}^*$ we denote $B^k = B_{\mathbb{R}^k}(0, 1)$ the unit euclidian ball and $\varphi_k(z) = \frac{1}{Vol(B^k)} 1_{B_k}(z)$.

lem7bis **Lemma 4.5** *Let ϵ be small. There exists $h_0 > 0, c_0, c_1 > 0$ and $M \in \mathbb{N}^*$ such that for all $h \in]0, h_0]$, one has*

$$T_h^M(x, dy) = \mu_h(x, dy) + c_0 h^{-Nd} \varphi_{Nd}(\frac{x - y}{c_1 h})dy \tag{4.30}$$ eq4.5

where for all $x \in \mathcal{O}_{N,\epsilon}$, $\mu_h(x, dy)$ is a positive Borel measure.

Proof. For $x, y \in \mathcal{O}_{N,\epsilon}$, we set $dist(x, y) = \sup_{1 \leq i \leq N} |x_i - y_i|$. For $N \geq 1$, let us denote by $K_{h,N}$ the kernel given in eq4.1 4.1. It is sufficient to prove the following: for ϵ small, there exists $h_0 > 0, c_0, c_1 > 0$ and $M(N) \in \mathbb{N}^*$ such that for all $h \in]0, h_0]$, one has for all non negative function f

$$K_{h,N}^{M(N)}(f)(x) \geq c_0 h^{-Nd} \int_{y \in \mathcal{O}_{N,\epsilon}, dist(y,x) \leq c_1 h} f(y)dy \tag{4.31}$$ g4

We first remark that it is sufficient to prove the weaker version: for all $x^0 \in \overline{\mathcal{O}}_{N,\epsilon}$, there exist $M(N, x^0), \alpha = \alpha(x^0) > 0$, $c_0 = c_0(x_0) > 0, c_1 = c_1(x_0) > 0, h_0 = h_0(x_0) > 0$ such that for all $h \in]0, h_0]$, all $x \in \mathcal{O}_{N,\epsilon}$ and all non negative function f

$$dist(x, x^0) \leq 2\alpha \Longrightarrow K_{h,N}^{M(N,x^0)}(f)(x) \geq c_0 h^{-Nd} \int_{y \in \mathcal{O}_{N,\epsilon}, dist(y,x) \leq c_1 h} f(y)dy \tag{4.32}$$ g5

Let us verify that 4.32 implies 4.31. Decreasing $\alpha(x_0)$ if necessary, we may assume that any set $\{dist(x,x^0) \leq 2\alpha(x_0)\}$ is contained in one of the open set U_l of lemma 4.2. There exists a finite set F such that $\overline{\mathcal{O}}_{N,\epsilon} \subset \cup_{x^0\in F}\{dist(x,x^0) \leq \alpha(x_0)\}$. Let $M(N) = sup_{x^0\in F} M(N,x_0)$, $c_i' = min_{x_0\in F} c_i(x_0)$ and $h_0' = min_{x_0\in F} h_0(x_0)$. One has to check that for any $x^0 \in F$ and any x with $dist(x,x^0) \leq \alpha(x^0)$, the right inequality in 4.32 holds true with $M(N) = M(N,x^0)+n$ in place of $M(N,x^0)$, and for some constants c_0, c_1, h_0 . Let U_l be such that $dist(x,x^0) \leq \alpha$ implies $x \in U_l$. Let j and $\Gamma_+(\nu,\delta)$ be given by lemma 4.2. Clearly, if f is non negative, one has

$$K_{h,N}^{M(N,x^0)+1}(f)(x) \geq \frac{1}{N}h^{-d}\int_{x+\sigma_j(z)\in\mathcal{O}_{N,\epsilon}} \varphi(z/h)K_{h,N}^{M(N,x^0)}(f)(x+\sigma_j(z))dz \tag{4.33}$$

g6

For $dist(x,x^0) \leq 2\alpha(x^0) - c_1'h/2$, and $|z| \leq c_1'h/2, z \in \Gamma_+(\nu,\delta)$, one has $dist(x+\sigma_j(z),x^0) \leq 2\alpha(x^0)$ and by ??, $x+\sigma_j(z) \in \mathcal{O}_{N,\epsilon}$. Moreover, $dist(y,x) \leq c_1'h/2 \implies dist(y,x+\sigma_j(z)) \leq c_1'h$. From 4.33 and 4.32 we thus get, with a constant C_δ depending only on the δ given by lemma 4.2, and for $h \leq h_0'$,

$$\begin{aligned} &dist(x,x^0) \leq 2\alpha(x^0) - c_1'h/2 \implies \\ &K_{h,N}^{M(N,x^0)+1}(f)(x) \geq \frac{C_\delta}{N}c_0'h^{-Nd}\int_{y\in\mathcal{O}_{N,\epsilon}, dist(y,x)\leq c_1'h/2} f(y)dy \end{aligned} \tag{4.34}$$

g7

By induction on n, we thus get

$$\begin{aligned} &dist(x,x^0) \leq 2\alpha(x^0) - c_1'h \implies \\ &K_{h,N}^{M(N,x^0)+n}(f)(x) \geq (\frac{C_\delta}{N})^n c_0'h^{-Nd}\int_{y\in\mathcal{O}_{N,\epsilon}, dist(y,x)\leq c_1'\frac{h}{2^n}} f(y)dy \end{aligned} \tag{4.35}$$

g7

Since n is bounded, we get the desired result with $h_0 = min(min_{x^0\in F}\alpha_{x^0}/c_1', h_0')$.

To complete the proof, let us show (4.32) by induction on N. The cas $N=1$ is obvious. Suppose that (4.32) holds for $N-1$ discs. Let $x^0 \in \overline{\mathcal{O}}_{N,\epsilon}$ being fixed. Thanks to Lemma 4.2, we can suppose that there exists an open neighbourhood U of x^0 a direction $\nu \in S^{d-1}$ and $\delta > 0$ such that (??) holds with $j=1$. Let us denote $x = (x_1, x')$ and

$$K_{h,N} = K_{h,N,1} + K_{h,N,>} \tag{4.36}$$

with

$$K_{h,N,1}f(x) = \frac{h^{-d}}{N}\int_{(y_1,x')\in\mathcal{O}_{N,\epsilon}} \varphi(\frac{x_1-y_1}{h})f(y_1,x')dy_1. \tag{4.37}$$

We also denote $G(\nu,\delta) = \Gamma_+(\nu,\delta) \cap \{|x_1| > \frac{\delta}{2}\}$. Then, we have the following

sublem7bis **Lemma 4.6** *For any $\delta' \in]0,\delta/2]$, there exists $C>0$, $\alpha>0$, $h_0>0$ and $r_0>0$ such $\forall r \in]0,r_0]$, $\forall h \in]0,h_0]$, $\forall x \in U\cap\mathcal{O}_{N,\epsilon}$, $\forall \tilde{x} \in x + h(G(\nu,\delta')\times B(0,r)^{N-1})$ with $\tilde{x}' \in \mathcal{O}_{N-1,\epsilon}$, we have $\tilde{x} \in \mathcal{O}_{N,\epsilon}$ and*

$$K_{h,N,>}f(\tilde{x}) \geq CK_{\alpha h,N-1}(f(\tilde{x}_1,.))(\tilde{x}') \tag{4.38}$$

eq:m1

for any non-negative function f. In particular, for all $M \in \mathbb{N}^*$, there exists C, r_0, h_0, a as above such that $\forall x \in U\cap\mathcal{O}_{N,\epsilon}$, $\forall \tilde{x} \in x + h(G(\nu,\delta')\times B(0,r)^{N-1})$, we have

$$K_{h,N,>}^M f(\tilde{x}) \geq CK_{\alpha h,N-1}^M(f(\tilde{x}_1,.))(\tilde{x}') \tag{4.39}$$

eq:m2

Proof. Inequality (4.39) is obtained easily from (4.38) by induction on M. To prove (4.38), we observe that for non-negative f and $\alpha \in]0,1[$ we have

$$K_{h,N,>}f(\tilde{x}) \geq \frac{h^{-d}}{N}\sum_{j=2}^{N}\int_{A_{j,\alpha,h}(\tilde{x})} f(\tilde{x}_1,\dots,y_j,\dots,\tilde{x}_N)dy_j \tag{4.40}$$

with $A_{j,\alpha,h}(\tilde{x}) = \{z \in \Omega,\ |\tilde{x}_j - z| < \alpha h \text{ and } \forall k \neq j,\ |\tilde{x}_k - z| > \epsilon\}$. Let $B_{j,\alpha,h}(\tilde{x}) = \{z \in \Omega,\ |\tilde{x}_j - z| < \alpha h \text{ and } \forall k \neq 1, j,\ |\tilde{x}_k - z| > \epsilon\}$. Then $A_{j,\alpha,h} \subset B_{j,\alpha,h}$ and we claim that for $\alpha, r > 0$ small enough and $\tilde{x} \in x + h(G(\nu,\delta') \times B(0,r)^{N-1})$ with $\tilde{x}' \in \mathcal{O}_{N-1,\epsilon}$, we have $B_{j,\alpha,h}(\tilde{x}) = A_{j,\alpha,h}(\tilde{x})$. Indeed, let $\tilde{x}_1 = x_1 + hu_1$ with $u_1 \in G(\nu,\delta')$ and $\tilde{x}' \in \mathcal{O}_{N-1,\epsilon}$ be such that $|\tilde{x}_j - x_j| < hr$. Then for $z \in B_{j,\alpha,h}(\tilde{x})$ we have

$$|\tilde{x}_1 - z| = |x_1 - x_j + hv_1| \tag{4.41}$$

with $v_1 = u_1 + \frac{x_j - \tilde{x}_j}{h} + \frac{\tilde{x}_j - z}{h}$. Taking α, r small enough (w.r.t. δ) it follows that $v_1 \in \Gamma_+(\nu,\delta)$. Consequently, Lemma 4.2 shows that $|\tilde{x}_1 - z| > \epsilon$ and hence $z \in A_{j,\alpha,h}(\tilde{x})$ (the same argument shows that $\tilde{x} \in \mathcal{O}_{N,\epsilon}$). Therefore,

$$K_{h,N,>}f(\tilde{x}) \geq \frac{h^{-d}}{N}\sum_{j=2}^{N}\int_{B_{j,\alpha,h}(\tilde{x})} f(\tilde{x}_1,\dots,y_j,\dots,\tilde{x}_N)dy_j = \frac{N}{N-1}K_{\alpha h,N-1}(f(\tilde{x}_1,.))(\tilde{x}') \tag{4.42}$$

and the proof of Lemma 4.6 is complete. □

Using this Lemma we can complete the proof of (4.32). Let $p \in \mathbb{N}$, $\alpha \in]0,\alpha_0]$ and $x \in \mathcal{O}_{N,\epsilon}$, then

$$\begin{aligned}
K_{h,N}^{p+1}f(x) &\geq K_{h,N,1}K_{h,N,>}^{p}f(x)\\
&\geq \frac{h^{-d}}{N}\int_{(z_1,x')\in\mathcal{O}_{N,\epsilon},z_1\in x_1+hG(\nu,\delta')} K_{h,N,>}^{p}f(z_1,x')dz_1\\
&\geq C\frac{h^{-d}}{N}\int_{(z_1,x')\in\mathcal{O}_{N,\epsilon},z_1\in x_1+hG(\nu,\delta')} K_{\alpha h,N-1}^{p}(f(z_1,.))(x')dz_1
\end{aligned} \tag{4.43}$$

thanks to Lemma 4.6. From the induction hypothesis we can choose $p \in \mathbb{N}$ so that

$$K_{h,N}^{p+1}f(x) \geq Ch^{-Nd}\int_{(z_1,x')\in\mathcal{O}_{N,\epsilon},z_1\in x_1+hG(\nu,\delta)}\int_{|x'-y'|<\alpha h,y'\in\mathcal{O}_{N-1,\epsilon}} f(z_1,y')dy'dz_1 \tag{4.44}$$

Hence, for any $\beta \in]0,1]$ we get

$$K_{h,N}^{p+2}f(x) \geq K_{h,N}^{p+1}K_{h,N,1}f(x) \geq Ch^{-Nd}\int_{D_{\alpha,\beta,h}(x)} f(y_1,y')\gamma_h(x,y_1)dydy' \tag{4.45}$$

with

$$D_{\alpha,\beta,h}(x) = \{y \in \mathcal{O}_{N,\epsilon},\ |x'-y'| < \alpha h, |x_1 - y_1| < \beta h\} \tag{4.46}$$

and

$$\gamma_h(x,y_1) = h^{-d}\int_{(z_1,x')\in\mathcal{O}_{N,\epsilon},z_1\in x_1+hG(\nu,\delta')} 1_{|z_1-y_1|<h}dz_1 \tag{4.47}$$

We have to show that γ_h is bounded from below by a positive constant, uniformly with respect to (x, y_1). For this purpose, we observe that for $y \in D_{\alpha,\beta,h}(x)$ and β, δ' small enough we have

$$\gamma_h(x, y_1) \geq Ch^{-d} \int_{(z_1,x')\in\mathcal{O}_{N,\epsilon}, z_1\in x_1+hG(\nu,\delta')} dz_1 = \int_{|u|<\alpha, u\in G(\nu,\delta')} 1_{(x_1+hu,x')\in\mathcal{O}_{N,\epsilon}} du \tag{4.48}$$

Using again Lemma 4.2, we get

$$\gamma_h(x, y_1) \geq C \int_{|u|<\alpha, u\in G(\nu,\delta')} du = C_0 > 0. \tag{4.49}$$

Plugging this lower bound into (4.45), we obtain

$$K_{h,N}^{p+2} \geq Ch^{-Nd} \int_{D_{\alpha,\beta,h}(x)} f(y)dy \tag{4.50}$$

and the proof of (4.32) is complete. This achieves the proof of lemma 4.5. $\square$

By proposition 4.1, we can consider the Neumann Laplacian $|\Delta|_N$ on $\mathcal{O}_{N,\epsilon}$ defined by

$$|\Delta|_N = -\frac{\alpha_d}{2N}\Delta,\ D(|\Delta|_N) = \{u \in H^1(\mathcal{O}_{N,\epsilon}),\ -\Delta u \in L^2(\mathcal{O}_{N,\epsilon}),\ \partial_n u|_{\partial\mathcal{O}_{N,\epsilon}} = 0\} \tag{4.51}$$

We still denote $0 = \nu_0 < \nu_1 < \nu_2 < \ldots$ the spectrum of $|\Delta|_N$ and m_j the multiplicity of ν_j. Our main result is the following.

Theorem 4.1 *Let $N \geq 2$ be fixed. Let $\epsilon > 0$ be small enough such that proposition 4.1 and lemma 4.5 holds true. Let $R > 0$ be given and $\beta > 0$ such that $\nu_{j+1} - \nu_j > 2\beta$ for all j such that $\nu_{j+2} \leq R$.*

There exists $h_0 > 0$, $\delta_0 \in]0, 1/2[$ and constants $C_i > 0$ such that for any $h \in]0, h_0]$, the following holds true.

i) The spectrum of T_h is a subset of $[-1 + \delta_0, 1]$, 1 is a simple eigenvalue of T_h, and $Spec(T_h) \cap [1 - \delta_0, 1]$ is discrete. Moreover,

$$\begin{aligned} &Spec(\frac{1-T_h}{h^2}) \cap]0, R] \subset \cup_{j\geq1}[\nu_j - \beta, \nu_j + \beta] \\ &\sharp Spec(\frac{1-T_h}{h^2}) \cap [\nu_j - \beta, \nu_j + \beta] = m_j \quad \forall \nu_j \leq R \end{aligned} \tag{4.52}$$

and for any $0 \leq \lambda \leq \delta_0 h^{-2}$, the number of eigenvalues of T_h in $[1 - h^2\lambda, 1]$ (with multiplicity) is bounded by $C_1(1 + \lambda)^{dN/2}$.

ii) The spectral gap $g(h)$ satisfies

$$\lim_{h\to0^+} h^{-2}g(h) = \nu_1 \tag{4.53}$$

and the following estimate holds true for all integer n

$$sup_{x\in\mathcal{O}_{N,\epsilon}} \|T_h^n(x, dy) - \frac{dy}{Vol(\mathcal{O}_{N,\epsilon})}\|_{TV} \leq C_4 e^{-ng(h)} \tag{4.54}$$

The rest of this section is devoted to the proof of theorem 4.1.

Let $\mu_h(x,dy)$ be given by 4.30 and $\mu_h(f)(x)=\int_{\mathcal{O}_{N,\epsilon}} f(y)\mu_h(x,dy)$. Thanks to the positivity of $\mu_h(x,dy)$, using the Markov property of T_h^M and Lipschitz-continuity of the boundary, we get for some $\delta_0'>0$ independant on $h>0$ small enough

$$\|\mu_h\|_{L^\infty,L^\infty}\leq 1-\inf_{x\in\mathcal{O}_{N,\epsilon}}\int_{\mathcal{O}_{N,\epsilon}} c_0h^{-Nd}\varphi_{Nd}(\frac{x-y}{c_1h})dy<1-\delta_0' \tag{4.55}$$

Since by 4.30 μ_h is selfadjoint on $L^2(\mathcal{O}_{N,\epsilon})$, we get also

$$\|\mu_h\|_{L^1,L^1}\leq 1-\delta_0' \tag{4.56}$$

and by interpolation it follows that $\|\mu_h\|_{L^2,L^2}\leq 1-\delta_0'$. In particular the essential spectrum of T_h^M is contained in $[0,1-\delta_0']$ so that $\sigma_{ess}(T_h)\subset[0,1-2\delta_0]$ with $2\delta_0=1-(1-\delta_0')^{1/M}$. Thus $Spec(T_h)\cap[1-\delta_0,1]$ is discrete. Let us verify that decreasing $\delta_0>0$, we may also assume

$$Spec(T_h)\subset[-1+\delta_0,1]. \tag{4.57}$$

Thanks to the Markov property of T_h^M, to prove this, it suffices to find $M\in 2\mathbb{N}+1$ such that

$$\int_\Omega\int_\Omega(u(x)+u(y))^2T_h^M(x,dy)dx\geq\delta_0\|u\|_{L^2}^2 \tag{4.58}$$

for any $u\in L^2(\Omega)$. Moreover, thanks to the proof of Lemma 4.5 there exists $M\in\mathbb{N}$ such that for any $n\in\mathbb{N}$,

$$\int_\Omega\int_\Omega(u(x)+u(y))^2T_h^{M+n}(x,dy)dx\geq c_0(n)h^{-Nd}\int_{\Omega\times\Omega}(u(x)+u(y))^2\varphi_{Nd}(\frac{x-y}{c_1(n)h})dxdy. \tag{4.59}$$

Hence, (4.57) follows from (4.67) and (2.6).

Following the strategy of section 2 we put $\mathcal{O}_{N,\epsilon}$ in a large box $B=]-A/2,A/2[^{Nd}$ and thanks to proposition 4.1 we define an extension map $E:L^2(\mathcal{O}_{N,\epsilon})\to L^2(B)$ wich is also bounded from $H^1(\mathcal{O}_{N,\epsilon})$ into $H^1(B)$. We denote

$$\mathcal{E}_{h,k}(u)=\langle(1-T_h^k)u,u\rangle_{L^2(\mathcal{O}_{N,\epsilon})} \tag{4.60}$$

and we define $\mathcal{E}_h$ as in section 2. Moreover the identities (2.11), (2.12) remain true with obvious modifications.

Lemma 4.7 *There exist $C_0,h_0>0$ such that the following holds true for any $h\in]0,h_0]$ and any $u\in L^2(\mathcal{O}_{N,\epsilon})$*

$$\mathcal{E}_h(E(u))\leq C_0(\mathcal{E}_{h,M}(u)+h^2\|u\|_{L^2}^2) \tag{4.61}$$

Proof. Thanks to Lemma 2.2 we have

$$\mathcal{E}_h(E(u))\leq C_0(\int_{\mathcal{O}_{N,\epsilon}\times\mathcal{O}_{N,\epsilon}}(u(x)-u(y))^2c_0h^{-Nd}\varphi_{Nd}(\frac{x-y}{c_1h})dydx+h^2\|u\|_{L^2(\mathcal{O}_{N,\epsilon})}) \tag{4.62}$$

Combined with (4.30) , this shows that

$$\mathcal{E}_h(E(u))\leq C_0(\int_{\mathcal{O}_{N,\epsilon}\times\mathcal{O}_{N,\epsilon}}(u(x)-u(y))^2T_h^M(x,dy)dx+h^2\|u\|_{L^2(\mathcal{O}_{N,\epsilon})}) \tag{4.63}$$

and the proof is complete. □

Lemma 4.8 *For any $0 \leq \lambda \leq \delta_0/h^2$, the number of eigenvalues of T_h in $[1-h^2\lambda, 1]$ (with multiplicity) is bounded by $C_1(1+\lambda)^{Nd/2}$. Moreover, any eigenfuntion $T_h(u) = \lambda u$ with $\lambda \in]1-\delta_0, 1]$ satisfies the bound*

$$\|u\|_{L^\infty} \leq C_2 h^{-Nd/2}\|u\|_{L^2} \tag{4.64}$$

Proof. Suppose that $T_h(u) = \lambda u$ with $\lambda \in [1-\delta_0, 1]$, then $T_h^M u = \lambda^M u$ and thanks to (4.30), we get

$$\|(\mu_h - \lambda^M)u\|_{L^\infty} = O(h^{-Nd/2}) \tag{4.65}$$

and estimate (4.64) follows from (4.55). On the other hand, thanks to Lemma 4.7, we can mimick the proof of Lemma 2.3 to get

$$\zeta_M(\lambda, h) \leq C(1+\lambda)^{Nd/2}. \tag{4.66}$$

Let $\zeta_k(\lambda, h)$ be the number of eigenvalues of T_h^k in the interval $[1-h^2\lambda, 1]$ for $h^2\lambda < \delta_0$. Then from 4.57, one has

$$\zeta_1(\lambda, h) = \zeta_k\left(\frac{1-(1-h^2\lambda)^k}{h^2}, h\right) \tag{4.67}$$

Combining (4.66) and (4.67) we get easily the anounced estimate. The proof of lemma 4.8 is complete. $\square$

The rest of the proof of Theorem 4.1 follows the strategy of sections 2 and 3. Using the spectral decomposition (2.39), (2.40) we get easily the estimates (2.46) and (2.48), and it remains to estimate $T_{h,1}^n$. Following the proof of Lemma 2.4, we can find $\alpha > 0$ small enough and $C > 0$ such that the following Nash inequality holds with $1/D = 2 - 4/p > 0$

$$\|u\|_{L^2}^{2+1/D} \leq Ch^{-2}((\mathcal{E}_{h,M}(u) + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/D}, \quad \forall u \in E_\alpha \tag{4.68}$$

From this inequality, we deduce that for $k \geq h^{-2}$,

$$\|T_{1,h}^{kM}\|_{L^\infty, L^\infty} \leq Ce^{-kMg(h)}. \tag{4.69}$$

and this implies for $k \geq h^{-2}$, since the contributions of $T_{2,h}^{kM}, T_{3,h}^{kM}$ are neglectible,

$$\|T_h^{kM}\|_{L^\infty, L^\infty} \leq C' e^{-kMg(h)}. \tag{4.70}$$

As T_h is bounded by 1 on L^∞ we can replace kM by $n \geq h^{-2}$ in (4.70) and (4.54) is proved. Assertion (4.53) is an obvious consequence of (4.52).The proof of (4.52) is the same as the one of Theorem 1.2. Thus, the following lemma will end the proof of theorem 4.1.

Lemma 4.9 *Let $\theta \in C^\infty(\overline{\mathcal{O}}_{N,\epsilon})$ be such that $supp(\theta) \cap \Gamma_{sing} = \emptyset$ and $\partial_n\theta|_{\Gamma_{reg}} = 0$. Then*

$$(1-T_h)\theta = h^2|\Delta|_N\theta + r, \quad \|r\|_{L^2} = O(h^{5/2}). \tag{4.71}$$

Proof. Let $\theta \in C^\infty(\overline{\mathcal{O}}_{N,\epsilon})$ be such that $supp(\theta) \cap \Gamma_{sing} = \emptyset$ and $\partial_n\theta|_{\Gamma_{reg}} = 0$ and denote $Q_h = 1 - T_h$. Then $Q_h = \frac{1}{N}\sum_{j=1}^N Q_{j,h}$ with

$$Q_{j,h}\theta(x) = \frac{h^{-d}}{Vol(B_1)}\int_\Omega 1_{|x_j - y| < h}\Pi_{k\neq j}1_{|x_k - y| > \epsilon}(f(x) - f(\pi_j(x) + \sigma_j(y)))dy \tag{4.72}$$

Let $\chi_0(x) = 1_{dist(x,\partial\mathcal{O}_{N,\epsilon})<2h}$. The same proof as in section 3 shows that

$$(1-\chi_0)Q_{j,h}\theta(x) = -\frac{\alpha_d}{2}h^2\partial_j^2\theta(x) + O_{L^\infty}(h^3) \tag{4.73}$$

so that

$$(1-\chi_0)Q_h\theta(x) = h^2|\Delta|_N\theta(x) + O_{L^2}(h^3). \tag{4.74}$$

We study $\chi_0 Q_h\theta$. As $\|\chi_0\|_{L^2} = O(h^{1/2})$ it suffices to show that $\|\chi_0 Q_h\theta\|_{L^\infty} = O(h^2)$. On the other hand, by Taylor expansion, we have

$$\chi_0 Q_{j,h}\theta(x) = -\frac{h\chi_0(x)}{Vol(B_1)}\int_{|z|<1}\Pi_{k\neq j}1_{|x_j+hz-x_k|>\epsilon}1_{x_j+hz\in\Omega}z.\partial_j\theta(x)dy_j + O_{L^\infty}(h^2) \tag{4.75}$$

Hence, it suffices to show that

$$v(x) = \chi_0(x)\sum_{j=1}^{N}\int_{|z|<1}\Pi_{k\neq j}1_{|x_j+hz-x_k|>\epsilon}1_{x_j+hz\in\Omega}z.\partial_j\theta(x)dy_j \tag{4.76}$$

satisfies $\|v\|_{L^\infty} = O(h)$. Since $dist(support(\theta), \Gamma_{sing}) > 0$, there exists disjoints compact sets $F_l \subset \Gamma_{reg,ext,l}$, and $F_{i,j} \subset \Gamma_{reg,int,(i,j)}$ such that

$$support(\theta) \subset \cup_l\{x, dist(x, F_l) \leq 4h\} \cup_{i,j} \{x, dist(x, F_{i,j}) \leq 4h\}$$

If $x \in support(\theta)$ is in $\{x, dist(x, F_1) \leq 4h\}$, then the same parity arguments as in section 3 show that

$$v(x) = \chi_0(x)\int_{|z|<1, x_1+hz\in\Omega} z.\partial_1\theta(x)dz = O(h) \tag{4.77}$$

If $x \in support(\theta)$ is in $\{x, dist(x, F_{1,2}) \leq 4h\}$, then

$$v(x) = \chi_0(x)\int_{|z|<1} z.(\partial_1\theta(x)1_{|x_1+hz-x_2|>\epsilon} + \partial_2\theta(x)1_{|x_2+hz-x_1|>\epsilon})dz \tag{4.78}$$

and the result follows from $(x_1 - x_2).(\partial_1\theta - \partial_2\theta)(x) = 0(h)$ for $\{x, dist(x, F_{1,2}) \leq 4h\}$, since $\partial_n\theta$ vanish on the boundary $|x_1 - x_2| = \epsilon$. The proof of lemma 4.9 is complete. □

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